DISQUISITIONES NONNULLAE DE AEQUATIONE:

$$A_n f(x+nh) + A_{n-1} f(x+(n-1)h) + \dots + A_1 f(x+h) + A f(x) = \varphi(x)$$

INDEQUE CASUUM SINGULORUM DEDUCTIO.

DISSERTATIO
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QUAM

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AMPLISSIMI PHILOSOPHORUM ORDINIS

IN

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AD

SUMMOS IN PHILOSOPHIA HONORES

RITE CAPESSENDOS

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H. L. Q. C.

palam defendet

JOSEPHUS THEOPHILUS USTYMOWICZ,

POSNANIENSIS.

OPPONENTIBUS:

H. KRZESIŃSKI, CAND. PHIL.

F. FABER, CAND. PHIL.

C. FUCHS, DR. PHIL.

VRATISLAVIAE,

TYPIS HENRICI RICHTERI.

Disquisitiones nonnullae de aequatione:

$A_n f(x+nh) + A_{n-1} f(x+(n-1)h) + \ldots + A_1 f(x+h) + A f(x) = \varphi(x)$

indeque casuum singulorum deductio.

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§. 1.

Data aequatione generali:

$$(1)\ A_n f(x+nh) + A_{n-1} f(x+(n-1)h) + A_{n-2} f(x+(n-2)h) + \ldots + A_1 f(x+h) + A f(x) = \varphi(x)$$

patet duplicem viam atque rationem eam considerandi nobis offerri. Denotantibus enim $A, A_1, A_2, A_3 \ldots A_n$ quaslibet certam legem sequentes quantitates aut dari $\varphi(x)$ et quaeri $f(x)$ potest, aut dari quoque $f(x)$, inde etiam $f(x+h)$, $f(x+2h)$ etc. etc., quum h sit incrementum constans, et quaeri $\varphi(x)$. Quoniam vero $f(x)$ prae ceteris ex aequatione hac definire maximi est momenti, propositum sit:

Ex data aequatione (1) $f(x)$ in seriem secundum quotientes differentiales ipsius $\varphi(x)$ progredientem evolvere.

Ex theoremate Tayloriano proficiscitur:

$$A f(x = A f(x)$$

$$A_1 f(x+h) = A_1 f(x) + A_1 h f'(x) + A_1 \frac{h^2}{1.2} f''(x) + A_1 \frac{h^3}{1.2.3} f'''(x) + \ldots + A_1 \frac{h}{1.2\ldots k} f^k(x) + \ldots$$

$$A_2 f(x+2h) = A_2 f(x) + 2A_2 h f'(x) + 2^2 A_2 \frac{h^2}{1.2} f''(x) + 2^3 A_2 \frac{h^k}{1.2.3} f'''(x) + \ldots + 2^k A_2 \frac{h^k}{1.2\ldots k} f^k(x) + \ldots$$

$$A_3 f(x+3h) = A_3 f(x) + 3A_3 h f'(x) + 3^2 A_3 \frac{h^2}{1.2} f''(x) + 3^3 A_3 \frac{h^3}{1.2.3} f'''(x) + \ldots + 3^k A_3 \frac{h^k}{1.2\ldots k} f^k(x) + \ldots$$

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$$A_{n-1} f(x+(n-1)h) = A_{n-1} f(x) + (n-1) A_{n-1} h f'(x)$$
$$+ (n-1)^2 A_n \frac{h^2}{1.2} f''(x) + \ldots + (n-1)^k A_{n-1} \frac{h^k}{1.2\ldots k} f^k(x) + \ldots$$
$$A_n f(x+nh) = A_n f(x) + nA_n h f'(x) + n^2 A_n \frac{h^2}{1.2} f''(x) + \ldots$$
$$+ n^k A_n \frac{h^k}{1.2\ldots k} f^k(x) + \ldots$$

Additis his aequationibus, terminisque singulis secundum quotientes $f'(x)$, $f''(x)$, $f'''(x)$ etc. ordinatis erit, si brevitatis causa ponatur:

$$2)\ A + A_1 + A_2 + A_3 + \ldots\ldots + A_n = a_0$$
$$A_1 + 2A_2 + 3A_3 + \ldots\ldots + nA_n = a_1$$
$$\frac{1}{1.2}\left(A_1 + 2^2A_2 + 3^2A_3 + \ldots\ldots + n^2A_n\right) = a_2$$
$$\frac{1}{1.2.3}\left(A_1 + 2^3A_2 + 3^3A_3 + \ldots\ldots + n^3A_n\right) = a_3$$

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$$\frac{1}{k!}\left(A_1 + 2^kA_2 + 3^kA_3 + \ldots\ldots n^kA^k\right) = a^k$$

$$3)\ \varphi(x) = a_0 f(x) + a_1 h f'(x) + a_2 h^2 f''(x) + a_3 h^3 f'''(x) + \ldots\ldots + a_k h^k f^k(x) + \ldots\ldots$$

Quodsi aequationem hanc pedetentim differentiemus et unamquamque inde derivatam suo loco per $\alpha_1 h$, $\alpha_2 h^2$, $\alpha_3 h^3$ etc. multiplicemus, fiet:

$$\alpha_1 h \varphi'(x) = \alpha_1 a_0 h f'(x) + \alpha_1 a_1 h^2 f''(x) + \alpha_1 a_2 h^3 f'''(x)$$
$$+ \alpha_1 a_3 h^4 f''''(x) + \ldots + \alpha_1 a_{k-1} h^k f^k(x) + \ldots$$
$$\alpha_2 h^2 \varphi''(x) = \ldots\ldots\ldots \alpha_2 a_0 h^2 f''(x) + \alpha_2 a_1 h^3 f'''(x)$$
$$+ \alpha_2 a_2 h^4 f''''(x) + \ldots + \alpha_2 a_{k-2} h^k f^k(x) + \ldots$$
$$\alpha_3 h^3 \varphi'''(x) = \ldots\ldots\ldots \alpha_3 a_0 h^3 f'''(x)$$
$$+ \alpha_3 a_1 h^4 f''''(x) + \ldots + \alpha_3 a_{k-3} h^k f^k(x) + \ldots$$
$$\alpha_4 h^4 \varphi''''(x) = \ldots\ldots\ldots$$
$$\alpha_4 a_0 h_4 f''''(x) + \ldots + \alpha_4 a_{k-4} h^k f^k(x) + \ldots$$

. . . .

$$\alpha_k h^k \varphi^k(x) = \ldots\ldots\ldots \alpha^k a_0 h^k f^k(x) + \ldots$$

. . . .

ex quo prodit:

$$4)\ \varphi(x) + \alpha_1 h \varphi'(x) + \alpha_2 h^2 \varphi''(x) + \alpha_3 h^3 \varphi'''(x) + \alpha_4 h^4 \varphi''''(x)$$
$$+ \ldots + \alpha_k h^k \varphi^k(x) \ldots =$$
$$= a_0 f(x) + (a_1 + \alpha_1 a_0) h f'(x) + (a_2 + \alpha_1 a_1 + \alpha_2 a_0) h^2 f''(x)$$
$$+ (a_3 + \alpha_1 a_2 + \alpha_2 a_1 + \alpha_3 a_0) h^2 f'''(x) + \ldots$$

$$\ldots + (a_k + \alpha_1 a_{k-1} + \alpha_2 a_{k-2} + \ldots + \alpha_{k-1} a_1 + \alpha^k a_0)\, h^k f^k(x) + \ldots .$$

Si nunc α_1, α_2, α_3, α_4 etc. etc. ita determinentur, ut coëfficientes ipsius $f'(x)$, $f''(x)$ $f'''(x)$ etc. evanescant, aut si ponatur:

$$5)\quad \begin{aligned} &a_1 + \alpha_1 a_0 = 0 \\ &a_2 + \alpha_1 a_1 + \alpha_2 a_0 = 0 \\ &a_3 + \alpha_1 a_2 + \alpha_2 a_1 + \alpha_3 a_0 = 0 \\ &a_4 + \alpha_1 a_3 + \alpha_2 a_2 + \alpha_3 a_1 + \alpha_4 a_0 = 0 \\ &\ldots \end{aligned}$$

$$(5{,}a)\quad a_k + \alpha_1 a_{k-1} + \alpha_2 a_{k-2} + \ldots \alpha_{h-1} a_{k-h+1} + \alpha_h a_{k-h} + \ldots\ldots + \alpha_{k-h} a_h + \alpha_{k-h+1} a_{h-1} + \ldots, + \alpha_{k-1} a_1 + \alpha_k a_0 = 0$$

seriem quaesitam:

$$6)\quad a_0 f(x) = \varphi(x) + \alpha_1 h \varphi'(x) + \alpha_2 h^2 \varphi''(x) + \alpha_3 h^3 \varphi'''(x) + \ldots + a_k h^k \varphi^k(x) + \ldots$$

definitis α_1, α_2, α_3, α_4 etc. etc. nanciscimur.

Computatis ergo ex aequationibus (5) his literis invenimus:

$$7)\quad \alpha_1 = -\frac{a_1}{a_0}$$

$$\alpha_2 = -\frac{a_2}{a_0} + \frac{a_1^2}{a_0^2}$$

$$\alpha_3 = -\frac{a_3}{a_0} + \frac{2a_2 a_1}{a_0^2} - \frac{a_1^3}{a_0^3}$$

$$\alpha_4 = -\frac{a_4}{a_0} + \frac{2a_3 a_1 + a_2^2}{a_0^2} - \frac{3a_2 a_1^2}{a_0^3} + \frac{a_1^4}{a_0^4}$$

$$\alpha_5 = -\frac{a_5}{a_0} + \frac{2a_4 a_1 + 2a_3 a_2}{a_0^2} - \frac{3a_3 a_1^3 + 3a_2^2 a_1}{a_0^3} + \frac{4a_2^2 a_1}{a_0^3} - \frac{a_1^5}{a_0^5}$$

$$\alpha_6 = -\frac{a_6}{a_0} + \frac{2a_5 a_1 + 2a_4 a_2 + a_3^2}{a_0^2} - \frac{3a_4 a_1^2 + 6a_3 a_2 a_1 + a_2^3}{a_0^3} + \frac{4a_3 a_1^3 + 6a_4^2 a_1^2}{a_0^4} - \frac{5a_2 a_1^4}{a_0^5} + \frac{a_1^6}{a_0^6}$$

. . . .

et generaliter fiat:

$$(7{,}a)\quad \alpha_k = -\frac{\overset{1}{C}_k}{a_0} + \frac{\overset{2}{C}_k}{a_0^2} - \frac{\overset{3}{C}_k}{a_0^3} + \frac{\overset{4}{C}_k}{a_0^4} - \ldots + (-1)^h \frac{\overset{h}{C}_k}{a_0^h} + \ldots .$$

Jam vero sola fere observatione generalem quoque legem de valoribus

modo repertis derivare possumus, cujus ope quodlibet α_k colligi queat. Animadvertimus enim:

1, signa a negativo initium sumere et alternatim se excipere;

2, denominatores progredi in serie geometrica;

3, summam coëfficientium numericorum numeratorum singulorum ex theoremate Neutoniano deduci posse, ita ut e. g. in α_5 valore $(1+1)^4$, in α_6 valore $(1+1)^5$, et omnino in α_k valore $(1+1)^{k-1}$ unusquisque illorum justo loco constituatur;

4, summam vero exponentium terminorum singulorum cujusvis numeratoris aequam esse exponenti denominatoris respondentis;

5, summam indicum in quovis termino numeratoris, aequam esse indici α, ad quod ille numerator pertineat, inhaerenti.

Ut autem legem hanc accuratius designemus demonstremusque, quemadmodum quantitates $a_1, a_2, a_3 \ldots a_k$ potestatesque earum in singulis numeratoribus mutentur atque inter se conjungantur, omnem numeratorem litera C (ut 7,a jam indicatum) significare volumus, cui quum supra notam localem, infra vero indicem ipsius α apposuerimus, $\overset{h}{C}_k$ ex ordine h^{tum} numeratorem valoris α_k dabit. Perspicuum nempe est, numeratorem $\overset{h}{C}_k$ nonnisi aequatione (5,a) definiri posse, qui quum ad denominatorem $a_0{}^h$ sit referendus, is vero in α_k ex $a_0{}^{h-1} \times a_0$ sit ortus (quia a_0 factor est ipsius α_k aequ. 5,a), $\overset{h}{C}_k$ ex omnibus, qui ad $a_0{}^{h-1}$ pertineant, conflatur numeratoribus.

At eo res magis nobis patebit, si pro $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_{k-1}$ valores ipsorum ea forma, quam (aeq. 7) habent, in aeq. 5,a neglectis signis, substitutos esse cogitaverimus, unde facile intelligemus, primum in α_{k-1} talem terminum inveniri, cui denominator $a_0{}^{h-1}$ erit. Quia autem α_{k-1} per a_1 est multiplicatum (5,a), numarator vero, cui idem denominator $a_0{}^{h-1}$ respondet, nostra significatione $= \overset{h-1}{C}_{k-1}$ erit, partem jam aliquam numeratoris $\overset{h}{C}_k$, adhibita expressione $\overset{h-1}{C}_{k-1} \cdot a_1$ determinare possumus. Deinde in α_{k-2} terminus invenitur, qui denominatorem $a_0{}^{h-1}$ habebit, etenim α_{k-2} quum per a_2 multiplictum sit (5,a) pars altera numeratoris $\overset{h}{C}_k$ valore $\overset{h-2}{C}_{k-2} \cdot a_2$ definitur. Sin autem hac ratione reliquas quoque partes designare pergimus, terminum ultimum, qui eundem habebit denominatorem, ad α_{h-1} pertinere cognoscimus, id quod valores (7) etiam denotant; quoniam vero α_{h-1} per a_{k-h+1} est multiplicatum, ultima pars numeratoris $\overset{h}{C}_k$ formam $\overset{h-1}{C}_{h-1} \times a_{k-h+1}$ induet. Collectis ergo valoribus his liquet esse:

$$8)\ \overset{h}{C}_k = \overset{h-a}{C}_{k-1} \cdot a_1 + \overset{h-1}{C}_{k-2} \cdot a_2 + \overset{h-1}{C}_{k-3} \cdot a_3 + \ldots + \overset{h-1}{C}_h \cdot a_{k-h} + \overset{h-1}{C}_{h-1} \cdot a_{k-h+1}$$

cui denominator $(-1)^h a_0{}^h$ respondebit.

Quodsi h = 1 ponatur aeq. 8. abibit in:

9) $\overset{1}{C}_k = a_k$; posito h = 2 fiet ipsa:

$$\overset{2}{C}_k = \overset{1}{C}_{k-1} \cdot a_1 + \overset{1}{C}_{k-2} \cdot a_2 + \overset{1}{C}_{k-3} \cdot a_3 + \ldots\ldots + \overset{1}{C}_2 \cdot a_{k-2} + \overset{1}{C}_1 \cdot a_{k-1}$$

Quum autem sit secundum Num. 9:

$$\overset{1}{C}_{k-1} = a_{k-1}$$
$$\overset{1}{C}_{k-2} = a_{k-2}$$
$$\overset{1}{C}_{k-3} = a_{k-3}$$
$$\cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot$$
$$\overset{1}{C}_2 = a_2$$

$\overset{1}{C}_1 = a_1$ erit ergo:

$$\overset{2}{C}_k = a_{k-1} \cdot a_1 + a_{k-2} \cdot a_2 + a_{k-3} \cdot a_3 + \ldots + a_2 \cdot a_{k-2} + a_1 \cdot a_{k-1}$$

Posito porro h=3 ex aeq. 8 evadit:

$$\overset{3}{C}_k = \overset{2}{C}_{k-1} \cdot a_1 + \overset{2}{C}_{k-2} \cdot a_2 + \overset{2}{C}_{k-3} \cdot a_3 + \ldots + \overset{2}{C}_3 \cdot a_{k-3} + \overset{2}{C}_2 \cdot a_{k-2}$$

Quum autem sit:

$$\overset{2}{C}_{k-1} = a_{k-2} \cdot a_1 + a_{k-3} \cdot a_2 + \ldots\ldots + a_2 \cdot a_{k-3} + a_1 \cdot a_{k-2}$$
$$\overset{2}{C}_{k-2} = a_{k-3} \cdot a_1 + a_{k-4} \cdot a_3 + \ldots\ldots + a_2 \cdot a_{k-4} + a_1 \cdot {}_{k-3}$$
$$\overset{2}{C}_{k-3} = a_{k-4} \cdot a_1 + a_{k-5} \cdot a_2 + \ldots\ldots + a_2 \cdot a_{k-5} + a_1 \cdot a_{k-4}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\overset{2}{C}_4 = a_3 \cdot a_1 + a_2 \cdot a_2 + a_1 \cdot a_3$$
$$\overset{2}{C}_3 = a_2 \cdot a_1 + a_1 \cdot a_2$$

$\overset{2}{C}_2 = a_1 \cdot a_1$ erit ergo:

$$\overset{2}{C}_k = \left(a_{k-2} \cdot a_1 + a_{k-3} \cdot a_2 + \ldots\ldots + a_1 \cdot a_{k-2}\right) a_1$$
$$+ \left(a_{k-3} \cdot a_1 + a_{k-4} \cdot a_2 + \ldots\ldots + a_1 \cdot a_{k-3}\right) a_2$$
$$+ \left(a_{k-4} \cdot a_1 + a_{k-5} \cdot a_2 + \ldots\ldots + a_1 \cdot a_{k-4}\right) a_3$$
$$+ \ldots\ldots + \left(a_2 \cdot a_1 + a_1 \cdot a_2\right) a_{k-3} + a_1 \cdot a_1 \, a_{k-2}$$

Simili modo demonstrari potest $\overset{4}{C}_k$ quoque e terminis componi, quorum quisque 4 elementa continet, quorumque indices additi summam k praebent. Id quod de $\overset{4}{C}_k$ valet, per conclusionem de $\overset{5}{C}_k, \overset{6}{C}_k, \overset{7}{C}_k \ldots \overset{h}{C}_k$ quoque valere debet. Omissis ergo brevitatis causa literis $a_0, a_1, a_2, a_3 \ldots$ facile intelligi potest, $\overset{2}{C}_k$ combinationem binorum elementorum cum omnibus, quotiescunque offeruntur, iterationibus ad summam k dari, in qua k-1 summum est elementum. Quare $\overset{2}{C}_k$ ita definire licet:

$\overset{2}{C}_k = {}^k\overset{2}{K}\,(1, 2, 3, 4 \ldots (k-1))$, si K signum combinationis, 2 numerum elementorum, k autem summam significat. Quoniam in ${}^k\overset{3}{K}$ summum elementum $= k-2$, in ${}^k\overset{4}{K}$ idem $= k-3$, in ${}^k\overset{5}{K}$, $= k-4$ etc. etc. solum esse debet, ergo in ${}^k\overset{h}{K}$ ipsum $= k - (h\text{-}1)$ fiat necesse est. Itaque simili modo demonstrari potest esse:

$$\overset{3}{C}_k = {}^k\overset{3}{K}\,(1, 2, 8, 4 \ldots\ldots (k-2))$$
$$\overset{4}{C}_k = {}^k\overset{4}{K}\,(1, 2, 3, 4 \ldots\ldots (k-3))$$
$$\overset{5}{C}_k = {}^k\overset{5}{K}\,(1, 2, 3, 4 \ldots\ldots (k-4))$$

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. . . . . . et generaliter:

$$\overset{h}{C}_k = {}^k\overset{h}{K}\,(1, 2, 3, 4 \ldots\ldots (k-(h-1)))$$

Conservato ordine arithmographico patet esse:

$$\overset{h}{C}_k = 1,{}^{k-1}\overset{h-1}{K}\,(1,2,3,4\ldots,k-(h-1)) + 2,{}^{k-2}\overset{h-1}{K}\,(1, 2, 3 \ldots, k-h)$$
$$+ 3,{}^{k-3}\overset{h-1}{K}(1,2,3 \ldots, k-(h+1)) + 4,{}^{k-4}\overset{h-1}{K}\,1,2,3 \ldots k-(h+2))$$
$$+ \ldots + (k-h)\,,{}^{h}\overset{h-1}{K}\,(1, 2) + (k-(h-1)\,,{}^{h-1}\overset{h-1}{K}\,(1)$$

qua formula omnes numeratoris partes expeditissime exhiberi possunt.

Quum vero numerus omnium combinationum ad summam k, sive numerus singulorum terminorum $= \dfrac{(h\text{-}(h\text{-}1))\,(k\text{-}(h\text{-}2))\,(k\text{-}(h\text{-}3))\ldots(k\text{-}2)\,(k\text{-}1)}{1\,.\,2\,.\,3\,.\,4\,.\,.\,.\,(h\text{-}2)\,(h\text{-}1)}$ sit, summa quoque coëfficientium numericorum idem fit valor, qui una terminus h^{tus} evolutionis $(1+1)^{k-1}$ est; quare nota 3 et 5 §. 1 est demonstrata.

Ut autem congruentes simul terminos colligamus Fischeri [1]) methodum similibus in problematibus solvendis sequentes, omissis iterum literis a_0,

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[1]) Theorie der Dimensionszeichen nebst ihrer Anwendung auf verschiedene Materien aus der Analysis endlicher Grössen. Halle 1792.

$a_1, a_2, \ldots$ ponamus esse elementa dimensionis unius $= 1, 2, 3, 4, 5$ etc. atque quaeratur ($\overset{h}{C_k}$) valor signorum dimensionis incertae ordinis h^{ti} [1]). Quum igitur hoc in casu ex h elementis summa $(h + (k - h))$ effici debeat, in eo res vertitur, ut prima loca quantum fieri potest unitate expleantur, ultima vero numeris summis. Erit ergo:

$$\alpha,\ h + (k - h) = 1 + 1 + 1 + 1 + \ldots \text{(usque ad summam } h - 1) + k - (h - 1)$$

$$\beta \begin{cases} 1,\ h + (k - h) = 1 + 1 + 1 + \ldots \text{(usque ad summam } h - 2) + 2 + (k - h) \\ 2,\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 2) + 3 + (k - (h + 1)) \\ 3,\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 2) + 4 + (k - (h + 2)) \\ \ldots \\ \ldots \\ \ldots \\ x\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 2) + \frac{2 + (k - h)}{2} + \frac{2 + (k - h)}{2} \end{cases}$$

$$\gamma \begin{cases} 1,\ h + (k - h) = 1 + 1 + 1 + \ldots \text{(usque ad summam } h - 3) + 2 + 2 + (k - (h + 1)) \\ 2,\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 3) + 2 + 3 + (k - (h + 2)) \\ 3,\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 3) + 2 + 4 + (k - (h + 3)) \\ \ldots \\ \ldots \\ \ldots \\ x\ \text{-}\ \text{-}\ \text{-} \quad = 1 + 1 + 1 + \ldots (\text{-}\ \text{-}\ \text{-}\ h - 3) + 2 + \frac{2 + (k - (h + 1))}{2} + \frac{2 + (k - (h + 1))}{2} \end{cases}$$

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[2]) Fischerus ibidem T. I. pag. 7 seq. denotat dimensionis signum ordinis primi per $\overset{1}{I}, \overset{2}{I}, \overset{3}{I}$ etc. ita ut e. g. functio data $y = Ax^m + Bx^{m+v} + Cx^{m+2v} + Dx^{m+3v} + \ldots$ ejus ratione scribi possit: $y = \overset{1}{I}x^m + \overset{2}{I}x^{m+v} + \overset{3}{I}x^{m+2v} + \ldots$ Ordini secundo signum dat II quantitates duarum dimensionum exprimens; inscriptum vero n ($\overset{n}{II}$) summam eorum omnium productorum significat, quae e duobus factoribus signi dimensionis unius ita efficiuntur, ut summa amborum indicum $= n$ fiat, producta autem ea toties computantur, quoties factores eorum permutari possunt. Datis ergo signis ordinis primi $\overset{1}{I}, \overset{2}{I}, \overset{3}{I}$ etc. signa incertae dimensionis ordinis p ita definiuntur:

$\overset{p}{I^p}, \overset{p+}{I^p}, \overset{p+2}{I^p} \ldots \overset{np}{I^p}$, atque ordinis $p + q$:

$$\delta\left\{\begin{array}{ll}
1,\ h+(k-h) = & 1+1+1+\ldots\ (\text{usque ad summam } h-3) \\
 & +\,3+3+(k-(h+3)) \\
2,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-3) \\
 & +\,3+4+(k-(h+4)) \\
3,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-3) \\
 & +\,3+5+(k-(h+5)) \\
\ldots & \ldots \\
\ldots & \ldots \\
x,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-3) \\
 & +\,3+\frac{3+(k-(h+3))}{2}+\frac{3+(k-(h+3))}{2} \\
\ldots & \ldots \\
\ldots & \ldots \\
y,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-3) \\
 & +\,\frac{6+(k-(h+3))}{3}+\frac{6+(k-(h+3))}{3}+\frac{6+(k-(h+3))}{3}
\end{array}\right.$$

$$\varepsilon\left\{\begin{array}{ll}
1,\ h+(k-h) = & 1+1+1+\ldots\ (\text{usque ad summam } h-4) \\
 & +\,2+2+2+(k-(h+2)) \\
2,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-4) \\
 & +\,2+2+3+(k-(h+3)) \\
3,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-4) \\
 & +\,2+2+4+(k-(h+4)) \\
 & \text{etc.}\quad \text{etc.}\quad \text{etc.}
\end{array}\right.$$

$$\zeta\left\{\begin{array}{ll}
1,\ h+(k-h) = & 1+1+1+\ldots\ (\text{usque ad summam } h-5) \\
 & +\,2+2+2+2+(k-(h+3)) \\
2,\ -\ -\ - = & 1+1+1+\ldots\ (\ -\ -\ -\ h-5) \\
 & +\,2+2+2+3+(k-(h+4)) \\
 & \text{etc.}\quad \text{etc.}\quad \text{etc.}
\end{array}\right.$$

Quum vero ad α pertineat numerus transpositionis $\dfrac{1.2.3.4\ldots.h}{1.2.3.4\ldots.(h-1)}$

$$= h$$

ad β 1, 2, 3, 4 . . . . . . $\dfrac{1.2.3.4\ldots.h}{1.2.3\ldots.(h-2)}$

$$= h\,(h-1)$$

$(IP+IQ)^{p+q}$, $(IP+IQ)^{p+q+1}$, $(IP+IQ)^{p+q+2}$ . . . . . . $(IP+IQ)^{np+nq}$

Nostro in problemate igitur signo $IC^{h+[k-h]}$ usus esset. Ex his, quae modo commemoravimus, jam constat, ipsius signa dimensionis nihil aliud nisi combinationes ad summas certas repraesentare.

Quum quoque ad β x pertineat idem . . . $\frac{1.2.3.4....h}{1.2.3....(h-2)1.2}$

$= \frac{h(h-1)}{1.2}$

ad γ 2, 3, 4 . δ 2, 3, 4 . . - - $\frac{1.2.3.4....h}{1.2.3...(h-3)}$

$= h(h-1)(h-2)$

ad γ 1 et x ; δ 1 et x - - $\frac{1.2.3.4....h}{1.2.3....(h-3)1.2}$

$= \frac{h(h-1)(h-2)}{1.2}$

ad δ, y - - - $\frac{1.2.3.4....h}{1.2.3..(h-3)1.2.3}$

$= \frac{h(h-1)(h-2)}{1.2.3}$

ad ε 2, 3, 4 etc. - - - $\frac{1.2.3.4....h}{1.2.3...(h-4)1.2}$

$= \frac{h(h-1)(h-2)(h-3)}{1.2}$

ad ε 1 et x - - - $\frac{1.2.3.4....h}{1.2.3..3(h-4)1.2.3}$

$= \frac{h(h-1)(h-2)(h-3)}{1.2.3}$

. . . .
. . . .
. . . .

ad ζ 2, 3, 4 etc. - - - $\frac{1.2.3.4....h}{1.2.3...(h-5)1.2.3}$

$= \frac{h(h-1)(h-2)(h-3)(h-4)}{1.2.3}$

ad ζ 1 et x - - - $\frac{1.2.3.4....h}{1.2.3..(h-5)1.2.3.4}$

$= \frac{h(h-1)(h-2)(h-3)(h-4)}{1.2.3.4}$

etc. etc. etc.

recepta ergo iterum litera a similibusque aggregatis valoribus erit numerator non dependens ipsius α_k :

$$10)\ \overset{h}{C}_k = h\,a_1^{h-1}.\,a_{k-(h-1)} + h(h-1)\,a_1^{h-2}\left(a_2 . a_{k-h} + a_3 . a_{k-(h+1)} + ..\right) +$$

$$+ \frac{h(h-1)}{1.2} a_1^{h-2} . a^2_{\frac{2+k-h}{2}} +$$

*

$$+ h(h-1)(h-2)\, a_1^{h-3}\left(a_2 \cdot a_3 \cdot a_{k-(h+2)} + a_2 \cdot a_4 \cdot a_{k-(h+3)} + \dots\right) +$$

$$+ \frac{h(h-1)(h-2)}{1 \cdot 2}\, a_1^{h-3}\left(a_2^2 \cdot a_{k-(h+1)} + a_3^2 \cdot a_{a-(h+3)} + \dots\right) +$$

$$+ \frac{h(h-1)(h-2)}{1 \cdot 2 \cdot 3}\, a_1^{h-3} \cdot a^3_{\frac{6+(k-[h+3])}{3}} + \frac{h(h-1)(h-2)(h-3)}{1 \cdot 2}$$

$$a_1^{h-4}\left(a_2^2 \cdot a_3\, a_{k-(h+3)} + \dots\right) + \dots$$

etc. etc. etc.

cui denominator $(-1)^h a_0^h$ respondebit. Quoniam autem indices ultimorum locorum ita se excipiunt, ut semper minorem major sequatur, aut ad summum alter alteri exaequetur, atque unusquisque numerus integer fiat necesse est, ideo et ordo abservandus est, nec non inquirendum, utrum e.g. $2+k-h$, $3+(k-(h+3))$ etc. per duo, $6+(k-(h+3))$ etc. per tres et ita porro dividi possit, nec ne; postremum enim si acciderit proximus minor numerus dividuus sumendus est, cui tum antecedentes indices inserviunt. Quodsi ope formulae hujus omnis fractio, ideoque omne α dato k et h computari potest, series quesita quoque:

11) $a_0 f(x) = \varphi(x) + \alpha_1 h \varphi'(x) + \alpha_2 h^2 \varphi''(x) + \dots + \alpha_k h^k \varphi^k(x) + \dots$ cum coëfficientibus α_1, α_2, α_3, α_4 etc. etc. determinata est.

Datis omnino $\varphi(x)$ et A, A_1, A_2 . . . A_n, quae variabiles etiam esse queunt quantitates, infinite magnum derivari posse numerum serierum facile intelligitur, numerus vero terminorum, quibus $f(x)$ (aeq. 11) determinatur, e gradu ipsius $\varphi(x)$ pendebit, quod si fractio erit, cujus denominator variabilem habebit valorem, $f(x)$ nonnisi per seriem infinitam exprimi potest.

Sin autem Lagragium secuti essemus, alia methodo ad solvendum problema nostrum uteremur, quae, quum quaeratur integrale completum, pariter ad aeq. 11, atque ad aeq. 3 accomodari potest: utramque enim, si quotientes differentiales solum respiciamus, ejusdem esse generis liquet, quam quidem aequationem linearem, sive melius primi gradus, ordinisque incerti nominare placuit [1]). Propositum ergo sit:

„Aequationis 11^mae^ integrale completum invenire.“

Quem ad finem fac quantitatem variabilem n^tum^ gradum non excedere atque $\varphi(x)$ esse functionem ipsius x rationalem integram ita, ut quotientes diff. $\varphi^{n+1}(x)$, $\varphi^{n+2}(x)$, etc. $= 0$ fiant, consideratur igitur aequatio, si

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[1]) Lacroix: Traité du calcul différentiel et du calcul integral. Tome II. p. 254 in adnotatione: Le mot lineaire est impropre; il est relatif à la géométrie, et en l'appliquant aux équations, on a eu en vue la ligne droite, dans l'équation de laquelle l'ordonnée et l'abscisse également ne se trouvent, qu'au primier degré: on ne sauroit donc regarder comme linéaires des équations tellesque $dy + Py\,dx = Q\,dx$, qui appartiennent le plus souvent à des courbes transcendantes.

dividatur ipsa per $\alpha_n h^n$, atque ponatur $\varphi(x) = y$, $\frac{a_0}{\alpha_n h^n} f(x) = V$, coëfficientes autem designentur ex ordine per P, Q, R . . . T, U, haec:

$$12) \frac{d^n y}{dx^n} + P \frac{d^{n-1} y}{dx^{n-1}} + Q \frac{d^{n-2} y}{dx^{n-2}} + R \frac{d^{n-3} y}{dx^{n-3}} + \ldots + T \frac{dy}{dx} + Uy = V$$

cujus forma est usitatissima. Jam vero Lagrangius demonstravit, aequationis (12) integrale completum inveniri posse, dummodo n valores particulares ipsius y cogniti sint, qui aequationi

$$13) \frac{d^n y}{dx^n} + P \frac{d^{n-1} y}{dx^{n-1}} + Q \frac{d^{n-2} y}{dx^{n-2}} + \ldots + T \frac{dy}{dx} + Uy = 0$$

satisfaciant [1]). Sint enim $y_1, y_2, y_3 \ldots y_n$ illi valores particulares, erit integrale completum aequationis (13)

$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \ldots + C_n y_n$, quod quoque aequationi (12) convenit, siquidem $C_1, C_2, C_3 \ldots C_n$, ut Lagrangius ostendit, tanquam functiones ipsius x considerantur, quae una cum variabili hac variantur. Quare si paulatim derivetur $\frac{dy}{dx}, \frac{d^2 y}{dx^2} \ldots \frac{d^n y}{dx^n}$, quum C_1, C_2, C_3 etc. simul mutentur, erit:

$$\frac{dy}{dx} = C_1 \frac{dy_1}{dx} + C_2 \frac{dy_2}{dx} + \ldots + C_n \frac{dy_n}{dx} + y_1 \frac{dC_1}{dx} + y_2 \frac{dC_2}{dx} + \ldots y_n \frac{dC_n}{dx}$$

Quoniam vero n quantitates ignotae determinentur necesse est, atque proposita quaestio unam solum conditionem praebet, ceteras $n - 1$ ad libitum sumere possumus. Itaque statuimus primum:

$y_1 \frac{dC_1}{dx} + y_2 \frac{dC_2}{dx} + y_3 \frac{dC_3}{dx} + \ldots + y_n \frac{dC_n}{dx} = 0$ esse, id quod dabit:

$$\frac{dy}{dx} = C_1 \frac{dy_1}{dx} + C_2 \frac{dy_2}{dx} + C_3 \frac{dy_3}{dx} + \ldots + C_n \frac{dy_n}{dx}$$

Quodsi valor hic iterum differentietur, erit:

$$\frac{d^2 y}{dx^2} = C_1 \frac{d^2 y_1}{dx^2} + C_2 \frac{d^2 y_2}{dx^2} + \ldots + C_n \frac{d^2 y_n}{dx^2} + \frac{dy_1}{dx} \cdot \frac{dC_1}{dx} + \frac{dy_2}{dx} \cdot \frac{dC_2}{dx} + \ldots + \frac{dy_n}{dx} \cdot \frac{dC_n}{dx}$$

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[1]) Aequatio haec integratur facta substitutione $y = e^{mx}$, si P,Q . . T,U constantes sunt, unde fit:

$$m^n + Pm^{n-1} + Qm^{n-2} + \ldots + Tm + U = 0$$

et si $m_1, m_2, m_3 \ldots m_n$, n valores sint, qui huic aequationi satisfaciunt, $e^{m_1 x}$, $e^{m_2 x}$, $e^{m_3 x} \ldots e^{m_n x}$ quoque satisfacere debent. Sint omnino n valores particulares $y_1, y_2, y_3 \ldots y_n$ qui aeq. 13 satisfaciunt, ponere possumus: $y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \ldots C_n y_n$, ubi C_1, C_2, C_3 etc. constantes sunt arbitrariae et peracta nostra substitutione, erit quoque $y = C_1 e^{m_1 x} + C_2^{m_2 x} + C_3^{m_3 x} + \ldots C_h e^{m_n x}$ expressio generalis integralis completi, siquidem $m_1, m_2, m_3 \ldots m_n$ radices sunt reales et impares; alii ubi intercedunt casus, aliae observentur solvendi rationes necesse est, quae legantur: Lacroix: Traité du calcul diff. et int. T. II. p. 315 sqq.

Posito rursum:

$$\frac{dy_1}{dx}\cdot\frac{dC_1}{dx}+\frac{dy_2}{dx}\cdot\frac{dC_2}{dx}+\frac{dy_3}{dx}\cdot\frac{dC_3}{dx}+\ldots+\frac{dy_n}{dx}\cdot\frac{dC_n}{dx}=0 \text{ fiet:}$$

$$\frac{d^2y}{dx^2}=C_1\frac{d^2y_1}{dx^2}+C_2\frac{d^2y_2}{dx^2}+C_3\frac{d^2y_3}{dx^2}+\ldots+C_n\frac{d^2y_n}{dx^2}$$

Si ita longius progrediamur, penultimas aequationes nanciscemur:

$$\frac{d^{n-2}y_1}{dx^{n-2}}\cdot\frac{dC_1}{dx}+\frac{d^{n-2}y_2}{dx^{n-2}}\cdot\frac{dC_2}{dx}+\ldots+\frac{d^{n-2}y_n}{dx^{n-2}}\cdot\frac{dC_n}{dx}=0$$

$$\frac{d^{n-1}y}{dx^{n-1}}=C_1\frac{d^{n-1}y_1}{dx^{n-1}}+C_2\frac{d^{n-1}y_2}{dx^{n-1}}+,\ldots+C_n\frac{d^{n-1}y_n}{dx^{n-1}}$$

unde denique evadit:

$$\frac{d^ny}{dx^n}=C_1\frac{d^ny_1}{dx^n}+C_2\frac{d^ny_2}{dx^n}+\ldots+C_n\frac{d^ny_n}{dx^n}+\frac{d^{n-1}y_1}{dx^{n-1}}\cdot\frac{dC_1}{dx}+\frac{d^{n-1}y_2}{dx^{n-1}}\cdot\frac{dC_2}{dx}+\ldots+\frac{d^{n-1}y_n}{dx^{n-1}}\cdot\frac{dC_n}{dx}$$

Substitutis his valoribus in aeq. (12) atque collectis simul terminis per $C_1, C_2, C_3 \ldots C_n$ multiplicatis prodit:

$$14)\left.\begin{array}{l} C_1\left(\frac{d^ny_1}{dx^n}+P\frac{d^{n-1}y_1}{dx^{n-1}}+Q\frac{d^{n-2}y_1}{dx^{n-2}}+\ldots+T\frac{dy_1}{dx}+Uy_1\right)\\ +C_2\left(\frac{d^ny_2}{dx^n}+P\frac{d^{n-1}y_2}{dx^{n-1}}+Q\frac{d^{n-2}y_2}{dx^{n-2}}+\ldots+T\frac{dy_2}{dx}+Uy_2\right)\\ +C_3\left(\frac{d^ny_3}{dx^n}+P\frac{d^{n-1}y_3}{dx^{n-1}}+Q\frac{d^{n-1}y_3}{dx^{n-2}}+\ldots+T\frac{dy_3}{dx}+Uy_3\right)\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ +C_n\left(\frac{d^ny_n}{dx^n}+P\frac{d^{n-1}y_n}{dn^{n-1}}+Q\frac{d^{n-2}y_n}{dx^{n-2}}+\ldots+T\frac{dy_n}{dx}+Uy_n\right)\\ +\frac{d^{n-1}y_1}{dx^{n-1}}\cdot\frac{dC_1}{dx}+\frac{d^{n-1}y_2}{dx^{n-1}}\cdot\frac{dC_2}{dx}+\frac{d^{n-1}y_3}{dx^{n-1}}\cdot\frac{dC_3}{dx}+\ldots+\\ \qquad+\frac{d^{n-1}y_n}{dx^{n-1}}\cdot\frac{dC_n}{dx} \end{array}\right\}=V$$

Quum vero $y_1, y_2, y_3 \ldots y_n$ particulares illi valores sint, qui aequationi (13) satisfaciunt, aequatio (14) abibit in hanc:

$$\frac{d^{n-1}y_1}{dx^{n-1}}\cdot\frac{dC_1}{dx}+\frac{d^{n-1}y_2}{dx^{n-1}}\cdot\frac{dC_2}{dx}+\frac{d^{n-1}y_3}{dx^{n-1}}\cdot\frac{dC_3}{dx}+\ldots+\frac{d^{n-1}y_n}{dx^{n-1}}\cdot\frac{dC_n}{dx}=V$$

Itaque ad determinandos n valores ignotos $\frac{dC_1}{dx}, \frac{dC_2}{dx} \ldots \frac{dC_n}{dx}$ nanciscimur n aequationes:

$$y_1\frac{dC_1}{dx}+y_2\frac{dC_2}{dx}+y_3\frac{dC_3}{dx}+\ldots+y_n\frac{dC_n}{dx}=0$$

$$\frac{dy_1}{dx}\cdot\frac{dC_1}{dx}+\frac{dy_2}{dx}\cdot\frac{dC_2}{dx}+\frac{dy_3}{dx}\cdot\frac{dC_3}{dx}+\ldots+\frac{dy_n}{dx}\cdot\frac{dC_n}{dx}=0$$

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$$\frac{d^{n-2}y_1}{dx^{n-2}}\cdot\frac{dC_1}{dx}+\frac{d^{n-2}y_2}{dx^{n-2}}\cdot\frac{dC_2}{dx}+\frac{d^{n-2}y_3}{dx^{n-2}}\cdot\frac{dC_3}{dx}+\ldots+\frac{d^{n-2}y_n}{dx^{n-2}}\cdot\frac{dC_n}{dx}=0$$

$$\frac{d^{n-1}y_1}{dx^{n-1}}\cdot\frac{dC_1}{dx}+\frac{d^{n-1}y_2}{dx^{n-1}}\cdot\frac{dC_2}{dx}+\frac{d^{n-1}y_3}{dx^{n-1}}\cdot\frac{dC_3}{dx}+\ldots+\frac{d^{n-1}y_n}{dx^{n-1}}\cdot\frac{dC_n}{dx}=V$$

ex quibus, quum $y_1, y_2, y_3 \ldots y_n$ valores sint noti, quotientes eos per x expressos invenire possumus, qui si sint:

$$\frac{dC_1}{dx}=X_1,\ \frac{dC_2}{dx}=X_2,\ \frac{dC_3}{dx}=X_3\ \ldots\ \frac{dC_n}{dx}=X_n \text{ fiet:}$$

$$C_1=\int X_1dx+E_1,\ C_2=\int X_2dx+E_2,\ C_3=\int X_3dx+E_3\ \ldots$$
$$C_n=\int X_ndx+E_n$$

atque integrale completum:

$$y=y_1\int X_1dx+E_1+y_2\int X_2dx+E_2+y_3\int X_3\,dx+E_3+\ldots+y_n\int X_ndx+E_n$$

Constantes $E_1, E_2, E_3 \ldots E_n$ propius designari nequeunt donec de integratione generali tractatur.

Eandem quoque methodum ad aeq. 3 nos usurpare potuisse videmus, si integrale f(x) definire conati essemus.

§. 2.

Si fit $a_0 = 0$ (6, §. 1) quantitates $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ etc. etc. supra commemorato modo determinari nequeunt; quum enim aequationes (5, §. 1.) $0 = 0$ evadant, fractiones pro iisdem valoribus (7, §. 1.) infinite magnae fiunt. Quare aequationem:

$$\varphi(x)+\alpha_1h\varphi'(x)+\alpha_2h^2\varphi''(x)+\ldots+\alpha_kh^k\varphi^k(x)+\ldots=a_0f(x)+$$
$$+(a_1+\alpha_1a_0)hf'(x)+(a_2+\alpha_1a_1+\alpha_2a_0)h^2f''(x)+$$
$$+(a_3+\alpha_1a_2+\alpha_2a_1+\alpha_3a_0)h^3f'''(x)+\ldots$$

iterum considerare liceat, quae accepto $a_0 = 0$ abibit in:

$$\varphi(x)+\alpha_1h\varphi'(x)+\alpha_2h^2\varphi''(x)+\ldots+\alpha_kh^k\varphi^k(x)\ldots=a_1hf'(x)+$$
$$+(a_2+\alpha_1a_1)h^2f''(x)+(a_3+\alpha_1a_2+\alpha_2a_1)h^3f'''(x)+\ldots$$

et si ponatur:

1) $a_2+\alpha_1a_1=0$
$a_3+\alpha_1a_2+\alpha_2a_1=0$
$a_4+\alpha_1a_3+\alpha_2a_2+\alpha_3a_1=0$

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$$a_{k+1}+\alpha_1a_k+\alpha_2a_{k-1}+\ldots+\alpha_ha_{k-h+1}+\ldots+\alpha_{k-h+1}a_h+\ldots+\alpha_ka_1=0$$

prodit series:

$$a_1 h f'(x) = \varphi(x) + \alpha_1 h \varphi'(x) + \alpha_2 h_2 \varphi''(x) + \ldots + \alpha_k h^k \varphi^k(x) + \ldots$$

et simplici integratione confecta:

$$a_1 h f(x) = \int \varphi(x)\, dx + \alpha_1 h \varphi(x) + \alpha_2 h^2 \varphi'(x) + \alpha_3 h^3 \varphi''(x) + \ldots + \alpha_k h^k \varphi^{k-1}(x) + \ldots \text{ aut}$$

$$(2)\ a_1 f(x) = \frac{1}{h} \int \varphi(x)\, dx + \alpha_1 \varphi(x) + \alpha_2 h \varphi'(x) + \ldots + \alpha_k h^{k-1} \varphi^{k-1}(x) + \ldots$$

cui constans arbitraria adjiciatur necesse est.

Sed valores $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ldots$ non modo ex aeq. (1. §. 2.), verum etiam ex aeq. (5. §. 1.) computari possunt, si notetur, expressiones recurrentes (5. §. 1.) transire in (1. §. 2.) siquidem unusquisque indicum literae a unitate evehatur. Quare valor quoque generalis $\overset{h}{C_k}$ (10. §. 1.) valebit, si modo a_0 in a_1, a_1 in a_2, a_2 in a_3 etc. mutetur.

Dato quoque $a_1 = 0$, ut $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ etc. definiantur, indices ipsius a iterum augeantur unitate necesse est, atque duplici integratione confecta, proficiscitur series:

$$a_2 f(x) = \frac{1}{h^2} \iint \varphi(x)\, dx^2 + \frac{\alpha_1}{h} \int \varphi(x)\, dx + \alpha_2 \varphi(x) + \alpha_3 h \varphi'(x) + \ldots + \alpha_k h^{k-2} \varphi^{k-2}(x) + \ldots$$

cui duae constantes arbitrariae tribuantur oportet.

Si omnino coëfficientes a_0, a_1, a_2, a_3 usque ad a_{k-1} evanescant, prodit series:

$$(3)\ a_k f(x) = \frac{1}{h^k} \int^k \varphi(x) dx^k + \frac{\alpha_1}{h^{k-1}} \int^{k-1} \varphi(x)\, dx^{k-1} + \frac{\alpha_2}{h^{k-2}} \int^{k-2} \varphi(x) dx^{k-2} + \ldots + \alpha_k \varphi(x) + \alpha_{k+1} h \varphi'(x) + \ldots$$

cui conveniunt k constantes arbitrariae.

Coëfficientes $\alpha_1, \alpha_2, \alpha_3$ etc. commutatis a_0 in a_k, a_1 in a_{k+1}, a_2 in a_{k+2} etc. etc. ex aeq. (5. §. 1.) determinari quoque possunt atque valor generalis $\overset{h}{C_k}$ hoc in casu etiam valebit, si modo commemorata commutatio valeat.

§. 3.

Priusquam ad singulares progrediamur casus, unam denique generalem quaestionem instituere volumus, eam quidem ex qua oritur: $f(x) = \Sigma^n \varphi(x)$; quem ad finem aequatio primitiva (1. §. 1.) in aequationem differentiarum immutatur. Quum enim sit:

$$A_n f(x + nh) = A_n f(x) + n A_n \Delta f(x) + \frac{n(n-1)}{1 \cdot 2} A_n \Delta^2 f(x) + \ldots + n A_n \Delta^{n-1} f(x) + A_n \Delta^n f(x)$$

$$A_{n-1} f(x + (n-1)h) = A_{n-1} f(x) + (n-1) A_{n-1} \Delta f(x) + \frac{(n-1)(n-2)}{1 \cdot 2} A_{n-1} \Delta^2 f(x) + \ldots + (n-1) A_{n-1} \Delta^{n-2} f(x) + A_{n-1} \Delta^{n-1} f(x)$$

$$A_{n-2}f(x+(n-2)h)=A_{n-2}f(x)+(n-2)A_{n-2}\Delta f(x)+\frac{(n-2)(n-3)}{1\,.\,2}A_{n-2}\Delta^2 f(x)+$$
$$\ldots+(n-2)\,A_{n-2}\Delta^{n-3}f(x)+A_{n-2}\Delta^{n-2}f(x)$$

. . . .

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. . . .

$$A_4\,f(x+4h)=A_4 f(x)+4A_4\Delta f(x)+\ldots+4A_4\Delta^3 f(x)+A_4\Delta^4 f(x)$$
$$A_3\,f(x+3h)=A_3 f(x)+3A_3\Delta f(x)+3A_3\Delta^2 f(x)+A_3\Delta^3 f(x)$$
$$A_2\,f(x+2h)=A_2 f(x)+2A_2\Delta f(x)+A_2\Delta^2 f(x)$$
$$A_1\,f(x+h)=A_1 f(x)+A_1\Delta f(x)$$
$$A\,f(x)=A\,f(x)$$

posito brevitatis causa:

$$(1)\quad A_n+A_{n-1}+\ldots+A_1+A=B$$
$$nA_n+(n-1)A_{n-1}+\ldots 2A_2+A_1=B_1$$
$$\frac{n(n-1)}{1\,.\,2}A_n+\frac{(n-1)(n-2)}{1\,.\,2}A_{n-1}+\ldots+6A_4+3A_3+A_2=B_2$$
$$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}A_n+\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}+\ldots+4A_4+A_3=B_3$$
$$\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}A_n+\frac{(n-1)(n-2)(n-3)(n-4)}{1\,.\,2\,.\,3\,.\,4}A_{n-1}+\ldots+5A_5+A_4=B_4$$

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$$\frac{n(n-1)}{1\,.\,2}A_n+(n-1)A_{n-1}+A_{n-2}=B_{n-2}$$
$$nA_n+A_{n-1}=B_{n-1}$$
$$A_n=B_n$$

erit igitur:

$$(2)\,B f(x)+B_1\Delta f(x)+B_2\Delta^2 f(x)+B_3\Delta^3 f(x)+\ldots+B_{n-2}\Delta^{n-2}f(x)+$$
$$+B_{n-1}\Delta^{n-1}f(x)+B_n\Delta^n f(x)=\varphi(x)$$

Et si fiet $B=0$, $B_1=0$, $B_2=0\ldots B_{n-2}=0$, $B_{n-1}=0$ nanciscimur:

$$B_n\Delta^n f(x)=\varphi(x)\text{ ergo}$$

$$(3)\quad f(x)=\frac{1}{B^n}\Sigma^n\varphi(x),\text{ si } B_n \text{ quantitas est constans.}$$

At erit tum quoque $a_{n-1}=0$, $a_{n-2}=0$, $a_{n-3}=0\ldots a_2=0, a_1=0$, $a_0=0$ quod ut demonstretur, aequationes conditionales:

$$B_{n-1}=nA_n+A_{n-1}=0$$
$$B_{n-2}=\frac{n(n-1)}{1\,.\,2}A_n+(n-1)A_{n-1}+A_{n-2}=0$$
$$B_{n-3}=\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}A_n+\frac{(n-1)(n-2)}{1\,.\,2}A_{n-1}+(n-2)A_{n-2}+A_{n-3}=0$$

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$$B_{n-h} = \frac{n(n-1)(n-2)\ldots(n-h+1)}{1.2.3\ldots h} A_n + \frac{(n-1)(n-2)\ldots(n-h+1)}{1.2.3\ldots(h-1)} A_{n-1} +$$
$$+ \frac{(n-2)(n-3)\ldots(n-h+1)}{1.2.3\ldots(h-2)} A_{n-2} + \ldots + \frac{(n-h+2)(n-h+1)}{1.2} A_{n-h+2}$$
$$+ (n-h+1) A_{n-h+1} + A_{n-h} = 0.$$

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considerantur, ex quibus $A_{n-1}, A_{n-2}, A_{n-3} \ldots A_{n-h} \ldots A_2, A_1$ per A_n exprimi possunt. Est enim:

$$A_{n-1} = - nA_n$$
$$A_{n-2} = \left(n(n-1) - \frac{n(n-1)}{1.2}\right) A_n = \left(\frac{2n(n-1)}{1.2} - \frac{n(n-1)}{1.2}\right) A_n =$$
$$= (2-1)\frac{n(n-1)}{1.2} A_n, \text{ quod dat:}$$
$$A_{n-2} = \left((-1)^2 - (1-1)^2\right)\frac{n(n-1)}{1.2} A_n = (-1)^2 \frac{n(n-1)}{1.2} A_n$$

Est insuper:

$$A_{n-3} = \left(-\frac{n(n-1)(n-2)}{1.2} + \frac{n(n-1)(n-2)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3}\right) A_n$$
$$= \left(-\frac{3n(n-1)(n-1)}{1.2.3} + \frac{3n(n-1)(n-2)}{1.2.3} + \frac{n(n-1)(n-2)}{1.2.3}\right) A_n$$
$$= \left(-3 + 3 - 1\right)\frac{n(n-1)(n-2)}{1.2.3} A_n \text{ quare:}$$
$$A_{n-3} = \left((-1)^3 + (1-1)^3\right)\frac{n(n-1)(n-2)}{1.2.3} A_n = (-1)^3 \frac{n(n-1)(n-2)}{1.2.3} A_n$$
$$A_{n-4} = \left((-1)^4 - (1-1)^4\right)\frac{n(n-1)(n-2)(n-3)}{1.2.3.4} A_n = (-1)^4 \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} A_n$$
$$A_{n-5} = \left((-1)^5 + (1-1)^5\right)\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} A_n$$
$$A_{n-5} = (-1)^5 \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} A_n$$

Atque generaliter demonstrari potest esse quoque:

$$A_{n-h} = (-1)^h \frac{n(n-1)(n-2)\ldots(n-h+1)}{1.2.3\ldots h} A_n$$

Prout in hac formula h alios atque alios valores adsciscit, aliud atque aliud A definitur. Sit enim $h = n - 3$ erit:

$$A_3 = (-1)^{n-3} \frac{n(n-1)(n-2)\ldots 5.4}{1.2.3\ldots(n-4)(n-3)} A_n = (-1)^{n-3} \frac{n(n-1)(n-2)}{1.2.3} A_n,$$

et posito $h = n - 2$

$$A_2 = (-1)^{n-2} \frac{n(n-1)}{1.2} A_n, \text{ atque tandem pro } h = n - 1$$
$$A_1 = (-1)^{n-1} nA_n$$

Intelligitur autem ex 2 §. 1 proficisci:

$$n!\,a_n = n^n A_n + (n-1)^n A_{n-1} + (n-2)^n A_{n-2} + \dots + 2^n A_2 + A_1$$

Quodsi pro $A_{n-1}, A_{n-2}, A_{n-3} \dots A_2, A_1$ modo definiti ponuntur valores, fiet:

$$n!\,a_n = A_n \left(n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n + \dots + (-1)^{n-2}\frac{n(n-1)}{1.2}2^n + (-1)^{n-1}n\right)$$

Brevitatis causa designemus:

$$\Phi_n^n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \dots + (-1)^{n-2}\frac{n(n-1)}{1.2}2^n + (-1)^{n-1}n$$

est etiam:

$$\Phi_n^n = n\left(n^{n-1} - (n-1)(n-1)^{n-1} + \frac{(n-1)(n-2)}{1.2}(n-2)^{n-1} - \dots + (-1)^{n-1}\right)$$

$$= n\left\{\begin{array}{l} n^{n-1} - \frac{n}{1}(n-1)^{n-1} + \frac{n(n-1)}{1.2}(n-2)^{n-1} - \dots + (-1)^{n-1}n \\ \quad + (n-1)^{n-1} - \frac{(n-1)}{1}(n-2)^{n-1} + \dots + (-1)^{n-2}(n-1) \end{array}\right\}$$

et adhibita nostra significatione fit:

$$4)\ \Phi_n^n = n\left(\Phi_n^{n-1} + \Phi_{n-1}^{n-1}\right)$$

Quum autem sit etiam:

$$\Phi_{n-1}^{n-1} = (n-1)\left\{\begin{array}{l} (n-1)^{n-2} - \frac{(n-1)}{1}(n-2)^{n-2} + \dots + (-1)^{n-1}(n-1) \\ \quad + (n-2)^{n-2} - \frac{(n-2)}{1}(n-3)^{n-2} + \dots + (-1)^{n-2}(n-2) \end{array}\right\}$$

erit secundum nostram significationem:

$$\Phi_{n-1}^{n-1} = (n-1)\left(\Phi_{n-1}^{n-2} + \Phi_{n-2}^{n-2}\right)$$

Atque simili modo facillime demonstrari potest, esse:

$$\Phi_{n-2}^{n-2} = (n-2)\left(\Phi_{n-2}^{n-3} + \Phi_{n-3}^{n-3}\right)$$

$$\Phi_{n-3}^{n-3} = (n-3)\left(\Phi_{n-3}^{n-4} + \Phi_{n-4}^{n-4}\right)$$

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$$\Phi_2^2 = 2\left(\Phi_2^1 + \Phi_1^1\right)$$

Quum autem omnis talis functio Φ_n^i, in qua $i < n$ est, fiat $= 0$ [1]), erit quoque $\Phi_n^{n-1} = 0$, $\Phi_{n-1}^{n-2} = 0$, $\Phi_{n-2}^{n-3} = 0 \ldots \Phi_2^1 = 0$, quia vero $\Phi_1^1 = 1$, substitutis pedetentim his valoribus prodit denique:

(5) $\Phi_n^n = n\,(n-1)\,(n-2)\,(n-3) \ldots 2 \,.\, 1$; qnare est

$n!\, a_n = n!\, A_n$

(6) $a_n = A_n$

Praeterea est (2. §. 1.):

$(n-1)!\, a_{n-1} = n^{n-1} A_n + (n-1)^{n-1} A_{n-1} + (n-2)^{n-1} A_{n-2} + \ldots + 2^{n-1} A_2 + A_1$

et si pro A_{n-1}, A_{n-2}, A_{n-3} etc. eorum valores substituantur, erit:

$$(n-1)!\, a_{n-1} = A_n \left(n^{n-1} - \frac{n}{1} (n-1)^{n-1} + \frac{n(n-1)}{1 \,.\, 2} (n-2)^{n-1} + \ldots + (-1)^{n-1} n \right)$$

$(n-1)!\; a_{n-1} = A_n\, \Phi_n^{n-1}$; quum vero sit $\Phi_n^{n-1} = 0$ ergo

$a_{n-1} = 0$

Pari ratione demonstratur esse:

$$(n-2)!\; a_{n-2} = A_n \left(n^{n-2} - \frac{n}{1} (n-1)^{n-2} + \frac{n(n-1)}{1 \,.\, 2} (n-2)^{n-2} + \ldots + (-1)^{n-1} n \right)$$

$(n-2)!\; a_{n-2} = A_n\, \Phi_n^{n-2}$; quia est $\Phi_n^{n-2} = 0$, erit

$a_{n-2} = 0$

Facile quoque intelligitur existere $a_{n-3} = 0$, $a_{n-4} = 0$, $a_{n-5} = 0$ etc. etc. quum exponentes functionis Φ_n^i magis magisque descendant, quae conditiones si observentur, nanciscimur secundum 3 §. 2. et 3 et 6. §. 3.

$$7)\ \frac{A_n}{B_n} \Sigma^n \varphi(x) = \Sigma^n \varphi(x) = \frac{1}{h^n} \int^n \varphi(x)\, dx^n + \frac{\alpha_1}{h^{n-1}} \int^{n-1} \varphi(x)\, dx^{n-1} +$$

$$+ \frac{\alpha_2}{h^{n-2}} \int^{n-2} \varphi(x)\, dx^{n-2} + \ldots + \alpha_n \varphi(x) + \alpha_{n+1}\, h \varphi'(x) + \ldots$$

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[1]) Innotessit enim esse:

$$\Delta^n f(x) = f(x + nh) - \frac{n}{1} f(x + (n-1)h) + \frac{n(n-1)}{1 \,.\, 2} f(x + (n-2)h) - \ldots \pm f(x)$$

et posito $f(x) = x^i$ erit:

$$\Delta^n x^i = (x + nh)^i - \frac{n}{1} (x + (n-1)h)^i + \frac{n(n-1)}{1 \,.\, 2} (x + (n-2)h)^i - \ldots \pm x^i$$

Quodsi est $i < n$ fit $\Delta^n x^i = 0$, quia differentia uno gradu semper minor est, quam functio primitiva. Erit ergo pro $i < n$

$0 = (x + nh)^i - \frac{n}{1} (x + (n-1)h)^i + \frac{n(n-1)}{1 \,.\, 2} (x + (n-2)h)^i - \ldots \pm x^i$; posito $x = 0$ et divisa aequatione per h^i

evadit $0 = n^i - \frac{n}{1} (n-1)^i + \frac{n(n-1)}{1 \,.\, 2} (n-2)^i - \ldots \pm n$ et igitur $\Phi_n^i = 0$ pro $i < n$.

quam ad seriem n constantes arbitrariae

$$\frac{C_1 x^{n-1}}{1.2.3..(n-1)} + \frac{C_2 x^{n-2}}{1.2.3..(n-2)} + \frac{C_3 x^{n-3}}{1.2.3..(n-3)} + .. + C_{n-1}x + C_n$$

integrationis causa adjiciantur necesse est.

Coefficientes vero α_1, α_2, α_3, α_4 etc. ex analogia §. 2 aequationibus:

$$8)\quad \begin{aligned} &a_{n+1} + \alpha_1 a_n = 0 \\ &a_{n+2} + \alpha_1 a_{n+1} + \alpha_2 a_n = 0 \\ &a_{n+3} + \alpha_1 a_{n+2} + \alpha_2 a_{n+1} + \alpha_3 a_n = 0 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

definiuntur, in quibus etiam supra allatae conditiones observandae sunt.

Est enim, ut jam demonstravimus (6), $a_n = A_n$; praetera fiet (2. §. 1.):

$$(n+1)!\, a_{n+1} = n^{n+1} A_n + (n-1)^{n+1} A_{n-1} + (n-2)^{n+1} A_{n-2} + \ldots + 2^{n+1} A_2 + A_1$$

et substitutis valoribus pro A_{n-1}, A_{n-2}, A_{n-3} etc. etc. habebimus:

$$(n+1)!\, a_{n+1} = A_n \left(n^{n+1} - \frac{n}{1}(n-1)^{n+1} + \frac{n(n-1)}{1.2}(n-2)^{n+1} + \ldots \right.$$
$$\left. + (-1)^{n-2}\frac{n(n-1)}{1.2}\, 2^{n+1} + (-1)^{n-1} n \right)$$

$(n+1)!\, a_{n+1} = A_n \Phi_n^{n+1}$, quare

$$9)\quad a_{n+1} = \frac{1}{(n+1)!} A_n \Phi_n^{n+1}$$

Quoniam autem functionem Φ in omni insequente a ad aliam atque aliam potestatem evehi demonstrabimus, generaliter considerare volumus:

$$\Phi_n^m = n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \frac{n(n-1)(n-2)}{1.2.3}(n-3)^m + ..$$

aut si Hindeburgiana designatione coefficientium binomialium utemur:

$$\Phi_n^m = n^m - {}^n\mathfrak{A}(n-1)^m + {}^n\mathfrak{B}(n-2)^m - {}^n\mathfrak{C}(n-3)^m + {}^n\mathfrak{D}(n-4)^m - \ldots$$

Ejecto in parte dextera n fiet:

$$\Phi_n^m = n\left(n^{m-1} - {}^{n-1}\mathfrak{A}(n-1)^{m-1} + {}^{n-1}\mathfrak{B}(n-2)^{m-1} - {}^{n-1}\mathfrak{C}(n-3)^{m-1} + ..\right)$$

et ergo:

$$\Phi_n^m = n \begin{pmatrix} n^{m-1} - {}^n\mathfrak{A}(n-1)^{m-1} + {}^n\mathfrak{B}(n-2)^{m-1} - {}^n\mathfrak{C}(n-3)^{m-1} + .. \\ + (n-1)^{m-1} - {}^{n-1}\mathfrak{A}(n-2)^{m-1} + {}^{n-1}\mathfrak{B}(n-3)^{m-1} - .. \end{pmatrix}^{1)}$$

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1) Est enim generaliter: ${}^n\mathfrak{M} - {}^{n-1}\overset{-1}{\mathfrak{M}} = \frac{n}{m}\,{}^{n-1}\overset{-1}{\mathfrak{M}} - {}^{n-1}\overset{-1}{\mathfrak{M}}$, ergo

$${}^n\mathfrak{M} - {}^{n-1}\overset{-1}{\mathfrak{M}} = \left(\frac{n}{m} - 1\right){}^{n-1}\overset{-1}{\mathfrak{M}} = \left(\frac{n-m}{m}\right){}^{n-1}\overset{-1}{\mathfrak{M}} = {}^{n-1}\mathfrak{M}$$

10) $\Phi_n^m = n\left(\Phi_n^{m-1} + \Phi_{n-1}^{m-1}\right)$[1] posito $m = n + k$ erit

$$\Phi_n^{n+k} = n\left(\Phi_n^{n+k-1} + \Phi_{n-1}^{n+k-1}\right)$$

Pari modo fit:

$$\Phi_{n-1}^{n+k-1} = (n-1)\left(\Phi_{n-1}^{n+k-2} + \Phi_{n-2}^{n+k-2}\right)$$

$$\Phi_{n-2}^{n+k-2} = (n-2)\left(\Phi_{n-2}^{n+k-3} + \Phi_{n-3}^{n+k-3}\right)$$

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[1]) Datis m et n tabula ad enumerandos valores numericos conformari atque ad libitum continuari potest, ut parvum hic specimen attulimus.

Quodsi pro $m = 7$ quoque numeros computare volumus, nanciscimur:

$\Phi_0^7 = 0$

$\Phi_1^7 = 1\left(1 + 0\right) = 1$

$\Phi_2^7 = 2\left(62 + 1\right) = 126$

$\Phi_3^7 = 3\left(540 + 62\right) = 1806$

$\Phi_4^7 = 4\left(1560 + 540\right) = 8400$ etc. etc.

Simul intelligitur inde pro $m < n$ esse $\Phi_n^m = 0$ atque pro $m = n$ fieri $\Phi_n^n = n!$

	m = 0	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 0	1	0	0	0	0	0	0
n = 1	0	1	1	1	1	1	1
n = 2	0	0	2	6	14	30	62
n = 3	0	0	0	6	36	150	540
n = 4	0	0	0	0	24	240	1560
n = 5	0	0	0	0	0	120	1800
n = 6	0	0	0	0	0	0	720

$$\Phi^{n}_{n-k} = (n-k)\left(\Phi^{n-1}_{n-k} + \Phi^{n-1}_{n-k-1}\right)$$

$$\Phi^{n-1}_{n-k-1} = (n-k-1)\left(\Phi^{n-2}_{n-k-1} + \Phi^{n-2}_{n-k-2}\right)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Phi^{k+2}_{2} = 2\left(\Phi^{k+1}_{2} + \Phi^{k+1}_{1}\right)$$

$$\Phi^{k+1}_{1} = 1\left(\Phi^{k}_{1} + \Phi^{k}_{0}\right)$$

Quoniam autem est $\Phi^{k}_{0} = 0$ (pag. 18 adn.) substitutis ergo paulatim his valoribus suppositoque $k < n$ procedit:

$$11)\ \Phi^{n+k}_{n} = n\Phi^{n+k-1}_{n} + n(n-1)\Phi^{n+k-2}_{n-1} + \ldots + n(n-1)\ldots(n-k)\Phi^{n-1}_{n-k} + \\ + n(n-1)\ldots(n-k-1)\Phi^{n-2}_{n-k-1} + \ldots + n(n-1)(n-2)\ldots 3.2\,\Phi^{k+1}_{2} + \\ + n(n-1)(n-2)\ldots 2.1\;\Phi^{k}_{1}$$

Posito tum $k = 1$ erit:

$$\Phi^{n+1}_{n} = n\,\Phi^{n}_{n} + n(n-1)\,\Phi^{n-1}_{n-1} + n(n-1)(n-2)\,\Phi^{n-2}_{n-2} + \ldots \\ + n(n-1)(n-2)\ldots 3.2.1\;\Phi^{1}_{1}$$

Quum vero secundum 5 sit $\Phi^{n-1}_{n-1} = (n-1)!$, $\Phi^{n-2}_{n-2} = (n-2)!$, $\Phi^{n-3}_{n-3} = (n-3)!$ etc. obtinebimus:

$$12)\ \Phi^{n+1}_{n} = n \times n! + n(n-1) \times (n-1)! + n(n-1)(n-2) \times (n-2)! + \ldots + n! \\ = n!\left(n + (n-1) + (n-2) + \ldots + 2 + 1\right)$$

quem valorem si in 9 substituerimus, fiet:

$$a_{n+1} = \frac{A_n}{n+1}\left(n + (n-1) + (n-2) + \ldots + 2 + 1\right)$$

aut summa in parenthesi per N_1 designata:

$$(9,\ a)\ a_{n+1} = \frac{A_n N_1}{n+1}$$

Ex 2 §. 1 habebimus porro:

$(n+2)!\,a_{n+2} = n^{n+2}A_n + (n-1)^{n+2}A_{n-1} + (n-2)^{n+2}A_{n-2} + \ldots + 2^{n+2}A_2 + A_1$ atque pro $A_{n-1}, A_{n-2} \ldots A_2, A_1$ illatis valoribus:

$$(n+2)!\,a_{n+2} = A_n\left(n^{n+2} - {}^{n}\mathfrak{A}\,(n-1)^{n+2} + {}^{n}\mathfrak{B}\,(n-2)^{n+2} - \ldots \right. \\ \left. + (-1)^{n-2}\,{}^{n}\mathfrak{B}\,2^{n+2} + (-1)^{n-1}\,{}^{n}\mathfrak{A}\right)$$

et adhibita nostra significatione:

$$13)\ a_{n+2} = \frac{1}{(n+2)!}\,A_n\Phi^{n+2}_{n}$$

Sed secundum 11, si ponatur $k = 2$, est:

$$\Phi_n^{n+2} = n\Phi_n^{n+1} + n(n-1)\Phi_{n-1}^{n} + n(n-1)(n-2)\Phi_{n-2}^{n-1} + \ldots$$
$$+ n(n-1)..2.1\Phi_1^2$$ et ope formulae 12 quum sit quoque:

$$\Phi_{n-1}^{n} = (n-1)!\Big((n-1) + (n-2) + .. + 2 + 1\Big)$$

$$\Phi_{n-2}^{n-1} = (n-2)!\Big((n-2) + (n-3) + .. + 2 + 1\Big)$$

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$$\Phi_2^3 = 2!\Big(2 + 1\Big)$$

$\Phi_1^2 = 1$ prodit:

$$14)\ \Phi_n^{n+2} = n!\Big[n\Big(n + (n-1) + \ldots + 2 + 1\Big) + (n-1)\Big((n-1) + (n-2) + .. + 2 + 1\Big) + .. + 3\Big(3 + 2 + 1\Big) + 2\Big(2 + 1\Big) + 1\Big]$$

Quare si valorem in parenthesi pesitum N_2 nominamus, erit:

$$(13,a)\ a_{n+2} = \frac{A_n N_2}{(n+1)(n+2)}$$

Simili modo datur:

$$(n+3)!\, a_{n+3} = n\ {}^{n+3}A_n + (n-1)\ {}^{n+3}A_{n-1} + (n-2)\ {}^{n+3}A_{n-2} + \ldots + 2\ {}^{n+3}A_2 + A_1$$

$$= A_n\Big(n^{n+3} - {}^{n}\mathfrak{A}(n-1)^{n+3} + {}^{n}\mathfrak{B}(n-2)^{n+3} - {}^{n}\mathfrak{C}(n-3)^{n+3} + .. + (-1)^{n-2}\ {}^{n}\mathfrak{B}\ 2^{n+3} + (-1)^{n-1}\ {}^{n}\mathfrak{A}\Big)$$

$$15)\ a_{n+3} = \frac{1}{(n+3)!} A_n\, \Phi_n^{n+3}$$

Quodsi in (11) $k = 3$ ponamus, habebimus:

$$\Phi_n^{n+3} = n\ \Phi_n^{n+2} + n(n-1)\Phi_{n-1}^{n+1} + n\ (n-1)\ (n-2)\ \Phi_{n-2}^{n} + \ldots + n\ (n-1)\ (n-2) \ldots 3\,.\,2\,.\,1\ \Phi_1^3$$

At vero Φ_n^{n+2} est jam (14) notum, ex quo deducitur:

$$\Phi_n^{n+1} = (n-1)!\Big[(n-1)\Big((n-1) + (n-2) + (n-3) + .. + 2 + 1\Big) + (n-2)\Big((n-2) + (n-3) + \ldots + 2 + 1\Big) + (n-3)\Big((n-3) + (n-4) + .. + 2 + 1\Big) + \ldots + 2\Big(2 + 1\Big) + 1\Big]$$

$$\Phi_{n-1}^{n} = (n-2)!\Big[(n-2)\Big((n-2)+(n-3)+\ldots+2+1\Big)+$$
$$+(n-3)\Big((n-3)+(n-4)+\ldots+2+1)\Big)+$$
$$+(n-4)\Big((n-4)+(n-5)+..+2+1\Big)+\ldots+2\Big(2+1\Big)+1\Big]$$
$$\Phi_{n}^{n-2} = (n-3)!\Big[(n-3)\Big((n-3)+(n-4)+\ldots+2+1\Big)+$$
$$+(n-4)\Big((n-4)+(n-5)+\ldots+2+1\Big)+$$
$$+(n-5)\Big((n-5)+(n-6)+..+2+1\Big)+\ldots+2\Big(2+1\Big)+1\Big]$$

etc. etc. etc.

quare confecta substitutione expositoque producto n! erit:

$$16)\ \Phi_{n}^{n+3} = n!\Big[n\Big(n\ (n+(n-1)+(n-2)+\ldots+2+1)+$$
$$+(n-1)(n-1)+(n-2)+..+2+1)+(n-2)\ (n-2)+(n-3)+..+2+1)+$$
$$+..+2(2+1)+1\Big)+(n-1)\Big((n-1)((n-1)+(n-2)+..+2+1)+$$
$$+(n-2)((n-2)+(n-3)+\ldots+2+1)+\ldots+2\ (2+1)+1\Big)+\ldots$$
$$+2\Big(2\ (2+1)+1\Big)+1\Big]$$

quem valorem in parenthesi si litera N_3 designemus, fiet:

$$15,\ a)\ a_{n+3} = \frac{A_n\ N_3}{(n+1)\ (n+2)\ (n+3)}$$

Itaque si in definiendis his valoribus longius procedamus, pari ratione invenimus:

$$(n+4)!\ a_{n+4} = A_n\Big(n^{n+4} - n\mathfrak{A}(n-1)^{n+4} + n\mathfrak{B}(n-2)^{n+4} - n\mathfrak{C}(n-3)^{n+4} + ..$$
$$+ (-1)^{n-2}\, n\mathfrak{B}\, 2^{n+4} + (-1)^{n-1}\, n\mathfrak{A}\Big)$$

$$17)\ a_{n+4} = \frac{1}{(n+4)!}\ A_n\ \Phi_{n}^{n+4}$$

$$(n+5)!\ a_{n+5} = A_n\Big(n^{n+5} - n\mathfrak{A}(n-1)^{n+5} + n\mathfrak{B}(n-2)^{n+5} - n\mathfrak{C}(n-3)^{n+5} + ..$$
$$+ (-1)^{n-2}\, n\mathfrak{B}\, 2^{n+5} + (-1)^{n-1}\, n\mathfrak{A}\Big)$$

$$18)\ a_{n+5} = \frac{1}{(n+5)!}\ A_n\ \Phi_{n}^{n+5}$$

et omnino:

$$(n+k)!\, a_{n+k} = A_n \Big(n^{n+k} - n\mathfrak{A}\,(n-1)^{n+k} + n\mathfrak{B}\,(n-2)^{n+k} + \dots + (-1)^{n-2}\, n\mathfrak{B}\; 2^{n+k} + (-1)^{n-1}\, n\mathfrak{A}$$

$$19)\quad a_{n+k} = \frac{1}{(n+k)!} A_n\, \Phi_n^{n+k}$$

Functiones vero Φ_n^{n+4}, Φ_n^{n+5} ... Φ_n^{n+k} per summas productorum similes quarum nonnulla exempla attulimus, exprimi posse liquet, neque nos latet lex, secundum quam (11) illae formantur.

Quodsi denique valores (6, 9, 13, 15 etc.) in aeq. 8 substituantur, ad determinandos coefficientes α_1, α_2, α_3 ... nanciscimur aequationes has:

$$20)\quad \frac{1}{(n+1)!}\, \Phi_n^{n+1} + \alpha_1 = 0$$

$$\frac{1}{(n+2)!}\, \Phi_n^{n+2} + \frac{\alpha_1}{(n+1)!}\, \Phi_n^{n+1} + \alpha_2 = 0$$

$$\frac{1}{(n+3)!}\, \Phi_n^{n+3} + \frac{\alpha_1}{(n+2)!}\, \Phi_n^{n+2} + \frac{\alpha_2}{(n+1)!}\, \Phi_n^{n+1} + \alpha_3 = 0$$

$$\vdots$$

in quas producta quoque nota pro functionibus Φ_n^{n+1}, Φ_n^{n+2}, Φ_n^{n+3} etc. recipi possunt. Non ignotum prorsus est, coefficientes α_1, α_2, α_3 .... singulari substitutione $\varphi(x) = e^x$ facta constitui etiam posse.

§. 4.

Si $n = 1$ ponatur, aequatio primitiva (1, §. 1.) transibit in:

$$1)\quad A_1\, f(x+h) + A\, f(x) = \varphi(x)$$

atque fiet (2, §. 1.):

$$2)\quad a_0 = A + A_1$$

$$a_1 = A_1$$

$$a_2 = \frac{A_1}{1\,.\,2}$$

$$a_3 = \frac{A_1}{1\,.\,2\,.\,3}$$

$$\vdots$$

$$a_k = \frac{A_1}{k!}$$

qui valores si in aeq. 5. §. 1. substituantur, prodit:

$$3)\quad A_1 + \alpha_1 (A + A_1) = 0$$

$$\frac{A_1}{1\,.\,2} + \alpha_1 A_1 + \alpha_2 (A + A_1) = 0$$

$$\frac{A_1}{1.2.3} + \alpha_1 \frac{A_1}{1.2} + \alpha_2 A_1 + \alpha_3 (A + A_1) = 0$$

. . . .

. . . .

$$\frac{A_1}{k!} + \alpha_1 \frac{A_1}{(k-1)!} + \alpha_2 \frac{A_1}{(k-2)!} + \ldots + \alpha_{k-1} A_1 + \alpha_k (A + A_1) = 0$$

unde fit secundum 7, §. 1.:

$$4)\ \alpha_1 = -\frac{A_1}{A + A_1}$$

$$\alpha_2 = -\frac{\frac{1}{1.2} A_1}{A + A_1} + \frac{A_1^2}{(A + A_1)^2}$$

$$\alpha_3 = -\frac{\frac{1}{1.2.3} A_1}{A + A_1} + \frac{\frac{2}{1.2} A_1^2}{(A + A_1)^2} - \frac{A_1^3}{(A + A_1)^3}$$

$$\alpha_4 = -\frac{\frac{1}{1.2.3.4} A_1}{A + A_1} + \frac{\left\{\frac{2}{1.2.3} + \left(\frac{1}{1.2}\right)^2\right\} A_1^2}{(A + A_1)^2} - \frac{\frac{3}{1.2} A_1^3}{(A + A_1)^3} + \frac{A_1^4}{(A + A_1)^4}$$

. . . .

. . . .

$$(4, a)\ \alpha_k = -\frac{\overset{1}{C}_k}{A + A_1} + \frac{\overset{2}{C}_k}{(A + A_1)^2} - \ldots + (-1)^h \frac{\overset{h}{C}_k}{(A + A)^h} + \ldots + (-1)^k \frac{\overset{k}{C}_k}{(A + A_1)^k}$$

Jam perspicuum est, omne α statim consignari posse, siquidem lex, quam singuli numeratorum coëfficientes numerici sequantur, constituta sit. At est 9, §. 1.:

5) $\overset{1}{C}_k = \frac{A_1}{k!}$ quare coëfficiens numericus $\frac{\overset{1}{C}_k}{A_1} = \frac{1}{k!}$ invenitur. Exsistit praeterea (ibidem):

$$\overset{2}{C}_k = \overset{1}{C}_{k-1} A_1 + \overset{1}{C}_{k-2} \frac{A_1}{1.2} + \ldots + \overset{1}{C}_2 \frac{A_1}{(k-2)!} + \overset{1}{C}_1 \frac{A_1}{(k-1)!}$$

quum vero sit:

$$\overset{1}{C}_{k-1} = \frac{A_1}{(k-1)!},\ \overset{1}{C}_{k-2} = \frac{A_1}{(k-2)!} \ldots \overset{1}{C}_2 = \frac{A_1}{1.2},\ \overset{1}{C}_1 = A_1$$

prodit:

$$\overset{2}{C}_k = \frac{A_1^2}{(k-1)!} + \frac{A_1^2}{1.2\,(k-2)!} + \ldots + \frac{A_1^2}{(k-2)!\,1.2} + \frac{A_1^2}{(k-1)!}$$

*

qui valor si dividatur per A_1^2 atque multiplicetur utrinque per k! fiet:

$$\frac{k!}{A_1^2}\overset{2}{C}_k = k + \frac{k(k-1)}{1 . 2} + \frac{k(k-1)(k-2)}{1 . 2 . 3} + \ldots + \frac{k(k-1)}{1 . 2} + k = 2^k - 2$$

Erit igitur alter coefficiens numericus:

$$6) \quad \frac{\overset{2}{C}_k}{A_1^2} = \frac{2^k - 2}{k!}, \text{ aut nostra significatione adhibita } = \frac{\Phi_2^k}{k!}$$

Numerator, qui secundum sequitur, est:

$$\overset{3}{C}_k = \overset{2}{C}_{k-1} A_1 + \overset{2}{C}_{k-2}\frac{A_1}{1 . 2} + \ldots . + \overset{2}{C}_3 \frac{A_1}{(k-3)!} + \overset{2}{C}_2 \frac{A_1}{(k-2)!}$$

et substitutis pro $\overset{2}{C}_{k-1}, \overset{2}{C}_{k-2}, \overset{2}{C}_{k-3} \ldots \overset{2}{C}_2$ valoribus ex (6) derivatis fiet:

$$\overset{3}{C}_k = A_1^3\left(\frac{\Phi_2^{k-1}}{(k-1)!} + \frac{\Phi_2^{k-2}}{1 . 2 (k-2)!} + \ldots + \frac{\Phi_2^3}{(k-3)!\, 1 . 2 . 3} + \frac{\Phi_2^2}{(k-2)!\, 1 . 2}\right)$$

aut

$$\frac{k!}{A_1^3}\overset{3}{C}_k = k\Phi_2^{k-1} + \frac{k(k-1)}{1 . 2}\Phi_2^{k-2} + \frac{k(k-1)(k-2)}{1 . 2 . 3}\Phi_2^{k-3} + \ldots .$$
$$+ \frac{k(k-1)(h-2)}{1 . 2 . 3}\Phi_2^3 + \frac{k(k-1)}{1 . 2}\Phi_2^2$$

Quodsi designata multiplicatio conficiatur, expressio partis dexterae in duas simpliciores M et N dividitur, quarum:

$$M = k\, 2^{k-1} + \frac{k(k-1)}{1 . 2} 2^{k-2} + \frac{k(k-1)(k-2)}{1 . 2 . 3} 2^{k-3} + \ldots + \frac{k(k-1)(k-2)}{1 . 2 . 3} 2^3 +$$
$$+ \frac{k(k-1)}{1 . 2} 2^2 = 3^k - 2^k - 2k - 1^k$$

$$N = -2\left(k + \frac{k(k-1)}{1 . 2} + \frac{k(k-1)(k-2)}{1 . 2 . 3} + \ldots + \frac{k(k-1)(k-2)}{1 . 2 . 3} + \frac{k(k-1)}{1 . 2}\right) =$$
$$= -2 . 2^k + 2k + 4 . 1^k$$

est, et igitur:

$M + N = \frac{k!}{A_1^3}\overset{3}{C}_k = 3^k - 3 . 2^k + 3 . 1^k = \Phi_3^k$, quare coefficiens numericus:

$$7) \quad \frac{\overset{3}{C}_k}{A_1^3} = \frac{\Phi_3^k}{k!} \text{ fit. —}$$

Est denique pari modo:

$$\overset{4}{C}_k = \overset{3}{C}_{k-1} . A_1 + \overset{3}{C}_{k-2} . \frac{A_1}{1 . 2} + \ldots + \overset{3}{C}_4 . \frac{A_1}{(k-4)!} + \overset{3}{C}_3 \frac{A_1}{(k-3)!}$$

$$\frac{k!}{A_1^4}\overset{4}{C}_k = k\Phi_3^{k-1} + \frac{k(k-1)}{1 . 2}\Phi_3^{k-2} + \frac{k(k-1)(k-2)}{1 . 2 . 3}\Phi_3^{k-3} + \ldots .$$
$$+ \frac{k(k-1)(k-2)(k-3)}{1 . 2 . 3 . 4}\Phi_3^4 + \frac{k(k-1)(k-2)}{1 . 2 . 3}\Phi_3^3$$

Si iterum designata multiplicatio conficiatur, valor dividitur in partes tres P, Q, R, quorum:

$$P = k\,3^{k-1} + \frac{k(k-1)}{1.2} 3^{k-2} + \frac{k(k-1)(k-2)}{1.2.3} 3^{k-3} + \ldots + \frac{k(k-1)(k-2)}{1.2.3} 3^3 =$$

$$= 4^k - 3^k - \frac{k(k-1)}{1.2} 3^2 - k.3 - 1^k$$

$$Q = -3\left(k\,2^{k-1} + \frac{k(k-1)}{1.2} 2^{k-2} + \frac{k(k-1)(k-2)}{1.2.3} 2^{k-3} + \ldots + \frac{k(k-1)(k-2)}{1.2.3} 2^3\right)$$

$$= -3.3^k + 3.2^k + 3\frac{k(k-1)}{1.2} 2^2 + 6k + 3.1^k$$

$$R = 3\left(k + \frac{k(k-1)}{1.2} + \frac{k(k-1)(k-2)}{1.2.3} + \ldots + \frac{k(k-1)(k-2)}{1.2.3}\right)$$

$$= 3.2^k - 3\frac{k(k-1)}{1.2} - 3k - 6.1^k$$

est, quare:

$$P + Q + R = \frac{k!}{A_1^4} \overset{4}{C}_k = 4^k - 4.3^k + 6.2^k - 4.1^k = \Phi_4^k,$$

quia est:

$$-\frac{k(k-1)}{1.2}\left(3^2 - 3.2^2 + 3.1^2\right) = -\frac{k(k-1)}{1.2} \Phi_3^2 = 0$$ (conf. 4, §. 3 adn. 1.)

Erit ergo coefficiens numericus termini quarti:

$$8)\quad \frac{\overset{4}{C}_k}{A_1^4} = \frac{\Phi_4^k}{k!}$$

Itaque in definiendis valoribus his progredientes suspicari possumus prodire prorsus:

$\frac{\overset{h}{C}_k}{A_1^h} = \frac{\Phi_h^k}{k!}$ qui valor omnino valebit, siquidem demonstretur, esse generaliter:

$$^{k}\mathfrak{A}\Phi_{h-1}^{k-1} + {}^{k}\mathfrak{B}\Phi_{h-1}^{k-2} + {}^{k}\mathfrak{C}\Phi_{h-1}^{k-1} + \ldots + {}^{k}\overset{-1}{\mathfrak{H}}\Phi_{h-1}^{h} + {}^{k}\mathfrak{H}\Phi_{h-1}^{h-1} = \Phi_h^k$$

Argumentationis igitur gratia valores singulos partis sinistrae dissolvamus, ita ut fiat:

$$^{k}\mathfrak{A}\Phi_{h-1}^{k-1} = {}^{k}\mathfrak{A}\Big((h-1)^{k-1} - {}^{h-1}\mathfrak{A}\,(h-2)^{k-1} + {}^{h-1}\mathfrak{B}\,(h-3)^{k-1} - \ldots$$

$$+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\,.\,2^{k-1} + (-1)^{h}\,{}^{h-1}\mathfrak{A}\,.\,1^{k-1}\Big)$$

$$^{k}\mathfrak{B}\Phi_{h-1}^{k-2} = {}^{k}\mathfrak{B}\Big((h-1)^{k-2} - {}^{h-1}\mathfrak{A}(h-2)^{k-2} + {}^{h-1}\mathfrak{B}\,(h-3)^{k-2} - \ldots$$

$$+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\,.\,2^{k-2} + (-1)^{h}\,{}^{h-1}\mathfrak{A}\,.\,1^{k-2}\Big)$$

$$
{}^{k}\mathfrak{C}\Phi_{h}^{k-3} = {}^{k}\mathfrak{C}\Big((h-1)^{k-3} - {}^{h-1}\mathfrak{A}\,(h-2)^{k-3} + {}^{h-1}\mathfrak{B}\,(h-3)^{k-3} - \dots
$$
$$
+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\,.\,2^{k-3} + (-1)^{h}\,{}^{h-1}\mathfrak{A}\,.\,1^{k-3}\Big)
$$

. . . .

. . . .

. . . .

$$
{}^{k}\overset{-1}{\mathfrak{H}}\Phi_{k-1}^{h} = {}^{k}\overset{-1}{\mathfrak{H}}\Big((h-1)^{h} - {}^{h-1}\mathfrak{A}\,(h-2)^{h} + {}^{h-1}\mathfrak{B}\,(h-3)^{h} - \dots
$$
$$
+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\,.\,2^{h} + (-1)^{h}\,{}^{h-1}\mathfrak{A}\,.\,1^{h}\Big)
$$

$$
{}^{k}\mathfrak{H}\Phi_{h-1}^{h-1} = {}^{k}\mathfrak{H}\Big((h-1)^{h-1} - {}^{h-1}\mathfrak{A}\,(h-2)^{h-1} + {}^{h-1}\mathfrak{B}\,(h-3)^{h-1} - \dots
$$
$$
+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\,.\,2^{h-1} + (-1)^{h}\,{}^{h-1}\mathfrak{A}\,.\,1^{h-1}\Big)
$$

Quodsi aequationes has addamus, summamque terminorum partis sinistrae $= S$ ponamus, atque valores partis dexterae secundum potestates $k-1$, $k-2$, $k-3$ etc. ordinemus, procedit:

$$
S = {}^{k}\mathfrak{A}\,(h-1)^{k-1} + {}^{k}\mathfrak{B}\,(h-1)^{k-2} + \dots + {}^{k}\overset{-1}{\mathfrak{H}}(h-1)^{h} + {}^{k}\mathfrak{H}\,(h-1)^{h-1}
$$
$$
- {}^{h-1}\mathfrak{A}\Big({}^{k}\mathfrak{A}(h-2)^{k-1} + {}^{k}\mathfrak{B}(h-2)^{k-2} + \dots + {}^{k}\overset{-1}{\mathfrak{H}}\,(h-2)^{h} + {}^{k}\mathfrak{H}\,(h-2)^{h-1}\Big)
$$
$$
+ {}^{h-1}\mathfrak{B}\Big({}^{k}\mathfrak{A}(h-3)^{k-1} + {}^{k}\mathfrak{B}(h-3)^{k-2} + \dots + {}^{k}\overset{-1}{\mathfrak{H}}(h-3)^{h} + {}^{k}\mathfrak{H}(h-3)^{h-1}\Big)
$$

. . . .

. . . .

$$
+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\Big({}^{k}\mathfrak{A}\,.\,2^{k-1} + {}^{k}\mathfrak{B}\,.\,2^{k-2} + \dots + {}^{k}\overset{-1}{\mathfrak{H}}\,.\,3^{h} + {}^{k}\mathfrak{H}\,.\,2^{h-1}\Big)
$$
$$
+ (-1)^{h}\,{}^{h-1}\mathfrak{A}\Big({}^{k}\mathfrak{A}\,.\,1^{k-1} + {}^{k}\mathfrak{B}\,.\,1^{k-2} + \dots + {}^{k}\overset{-1}{\mathfrak{H}}\,.\,1^{h} + {}^{k}\mathfrak{H}\,.\,1^{h-1}\Big)
$$

aut etiam:

$$
S = h^{k} - (h-1)^{k} - {}^{k}\overset{1}{\mathfrak{H}}\,(h-1)^{h-2} - \dots - {}^{k}\mathfrak{A}\,(h-1) - 1^{k}
$$
$$
- {}^{h-1}\mathfrak{A}\Big((h-1)^{k} - (h-2)^{k} - {}^{k}\overset{+1}{\mathfrak{H}}(h\ 2)^{h-2} - \dots - {}^{k}\mathfrak{A}\,(h-2) - 1^{k}\Big)
$$
$$
+ {}^{h-1}\mathfrak{B}\Big((h-2)^{k} - (h-3)^{k} - {}^{k}\overset{+1}{\mathfrak{H}}(h-3)^{h-2} - \dots - {}^{k}\mathfrak{A}(h-3) - 1^{k}\Big)
$$

. . . .

. . . .

$$
+ (-1)^{h-1}\,{}^{h-1}\mathfrak{B}\Big(3^{k} - 2^{k} - {}^{k}\overset{+1}{\mathfrak{H}}\,.\,2^{h-2} - \dots - {}^{k}\mathfrak{A}\,.\,2 - 1^{k}\Big)
$$
$$
+ (-1)^{h}\,{}^{h-1}\mathfrak{A}\Big(2^{k} - 1^{k} - {}^{k}\overset{+1}{\mathfrak{H}}\,.\,1^{h-2} - \dots - {}^{k}\mathfrak{A}\,.\,1 - 1^{k}\Big)
$$

Si nunc prima secunda et ultima verticalis concipiatur series, evenit:

$$h^k - \left({}^{h-1}\mathfrak{A} + 1\right)(h-1)^k + \left({}^{h-1}\mathfrak{A} + {}^{h-1}\mathfrak{B}\right)(h-2)^k - \left({}^{h-1}\mathfrak{B} + {}^{h-1}\mathfrak{C}\right)(h-3)^h + \ldots + (-1)^{h+1}\left({}^{h-1}\mathfrak{A} + 1\right) . 1^k$$

Quum vero sit:

$$- {}_k\overset{+1}{\mathfrak{H}}\Big((h-1)^{h-2} - {}^{h-1}\mathfrak{A}(h-2)^{h-2} + {}^{h-1}\mathfrak{B}(h-3)^{h-2} - \ldots + (-1)^{h-1}\, {}^{h-1}\mathfrak{B}\, 2^{h-2} + (-1)^h\, {}^{h-1}\mathfrak{A}\ 1^{h-2}\Big) = - {}_k\overset{+1}{\mathfrak{H}}\Phi^{h-2}_{h-1} = 0$$

$$- {}_k\overset{+2}{\mathfrak{H}}\Big((h-1)^{h-3} - {}^{h-1}\mathfrak{A}(h-2)^{h-3} + {}^{h-1}\mathfrak{B}(h-3)^{h-3} - \ldots + (-1)^{h-1}\, {}^{h-1}\mathfrak{B}\, 2^{h-3} + (-1)^h\, {}^{h-1}\mathfrak{A}\, 1^{h-3}\Big) = - {}_k\overset{+2}{\mathfrak{H}}\Phi^{h-3}_{h-1} = 0$$

omnesque sequentes verticales series evanescant (4. §. 3 adn.) quia exponens functionis Φ gradatim minuitur, restat:

$$S = h^k - \left({}^{h-1}\mathfrak{A} + 1\right)(h-1)^k + \left({}^{h-1}\mathfrak{A} + {}^{h-1}\mathfrak{B}\right)(h-2)^k - \left({}^{h-1}\mathfrak{B} + {}^{h-1}\mathfrak{C}\right)(h-3)^k + \ldots + (-1)^{h+1}\left({}^{h-1}\mathfrak{A} + 1\right)1^k$$

$$S = h^k - {}^{h}\mathfrak{A}\,(h-1)^k + {}^{h}\mathfrak{B}\,(h-2)^k - {}^{h}\mathfrak{C}\,(h-3)^k + \ldots + (-1)^{h+1}\, {}^{h}\mathfrak{A}\ 1^k\ ^{*)} = \Phi^k_h$$

Erit ergo revera:

$$9)\ {}^k\mathfrak{A}\Phi^{k-1}_{h-1} + {}^k\mathfrak{B}\Phi^{k-2}_{h-1} + {}^k\mathfrak{C}\Phi^{k-3}_{h-1} + \ldots + {}^k\overset{-1}{\mathfrak{H}}\Phi^{h}_{h-1} + {}^k\mathfrak{H}\Phi^{h-1}_{h-1} = \Phi^k_h$$

quare haud dubie fiet coefficiens numericus h^{tus}:

$$10)\ \frac{\overset{h}{C}_k}{A^h} = \frac{\Phi^k_h}{k!}$$

Si denique valores 5, 6, 7, 8 ... 10 etc. in (4,a) substituerimus, habebimus:

$$11)\ \alpha_k = \frac{1}{k!}\left(-\frac{\Phi^k_1 A_1}{A + A_1} + \frac{\Phi^k_2 A_1^2}{(A+A_1)^2} - \frac{\Phi^k_3 A_1^3}{(A+A_1)^3} + \ldots + (-1)^h \frac{\Phi^k_h A_1^h}{(A+A_1)^h} + \ldots + (-1)^k \frac{\Phi^k_k A_1^k}{(A+A_1)^k}\right)$$

ubi Φ^k_1, Φ^k_2, Φ^k_3, et omnino Φ^k_h dato h et k secundum allatam tabulam (pag. 20) enumerari possunt.

---

*) Quia est omnino $^{n-1}\mathfrak{M} + {}^{n-1}\overset{-1}{\mathfrak{M}} = \left(\frac{n-m}{m} + 1\right) {}^{n-1}\overset{-1}{\mathfrak{M}} = \frac{n}{m}\, {}^{n-1}\overset{-1}{\mathfrak{M}} = {}^{n}\mathfrak{M}$

Dato tandem aut $f(x)$ aut $\varphi(x)$, conservataque conditione $n = 1$ series ex 11 §. 1 proficiscens:

$$12)\quad (A + A_1)\, f(x) = \varphi(x) + \alpha_1 h \varphi' + \alpha_2 h^2 \varphi''(x) + \ldots$$

cum coefficientibus α_1, α_2, α_3, α_4 etc. penitus cognita est.

Quodsi nunc valores 4. §. 4. ad eundem revocamus denominatorem, prodit:

$$13)\quad \alpha_1 = -\frac{A_1}{A + A_1}$$

$$\alpha_2 = -\frac{A_1 A - A_1^2}{1.2\,(A + A_1)^2}$$

$$\alpha_3 = -\frac{A_1 A^2 - 4A_1^2 A + A_1^3}{1.2.3\,(A + A_1)^3}$$

$$\alpha_4 = -\frac{A_1 A^3 - 11 A_1^2 A^2 + 11 A_1^3 A - A_1^4}{1.2.3.4\,(A + A_1)^4}$$

$$\alpha_5 = -\frac{A_1 A^4 - 26 A_1^2 A^3 + 66 A_1^3 A^2 - 26 A_1^4 A + A_1^5}{1.2.3.4.5\,(A + A_1)^5}$$

$$\ldots\ldots\ldots\ldots$$

$$(13,a)\quad \alpha_k = -\frac{\mathfrak{A}_1 A_1 A^{k-1} - \mathfrak{A}_2 A_1^2 A^{k-2} + \mathfrak{A}_3 A_1^3 A^{k-3} - \ldots}{1.2.3.4\ldots(k-1)\,k\,(A + A_1)^k} \frac{+ (-1)^k \mathfrak{A}_{k-1} A_1^{k-1} A}{1.2.3.4\ldots} \frac{+ (-1)^{k+1} \mathfrak{A}_k A_1^k}{(k-1)\,k\,(A + A_1)^k}$$

qua dispositione adhibita Eulerus [1]) invenit et Laplace [2]) approbavit esse:

$$\mathfrak{A}_1 = 1^k$$

$$\mathfrak{A}_2 = 2^k - (k+1) \,.\, 1^k$$

$$\mathfrak{A}_3 = 3^k - (k+1)\, 2^k + \frac{(k+1)\,k}{1.2} \,.\, 1^k$$

$$\mathfrak{A}_4 = 4^k - (k+1)\, 3^k + \frac{(k+1)\,k}{1.2}\, 2^k - \frac{(k+1)\,k\,(k-1)}{1.2.3} \,.\, 1^k$$

$$\mathfrak{A}_5 = 5^k - (k+1)\, 4^k + \frac{(k+1)\,k}{1.2}\, 3^k - \frac{(k+1)\,k\,(k-1)}{1.2.3}\, 2^k + \ldots$$

$$\ldots\ldots\ldots\ldots$$

$$\mathfrak{A}_k = k^k - {}^{k+1}\mathfrak{A}\,(k-1)^k + {}^{k+1}\mathfrak{B}\,(k-2)^k - {}^{k+1}\mathfrak{C}\,(k-3)^k + \ldots$$
$$+ (-1)^{k+1}\; {}^{k+1}\overset{-1}{\mathfrak{K}} \,.\, 1^k$$

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1) Euleri: Institutiones calculi differentialis cum ejus usu in analysi infinitorum ac doctrina serierum. Cap. VII. pag. 482 sqq.

2) Legitur: Lacroix: Traité du calcul diff. et intg. Tom. III. pag. 107—110.

id quod ex nostra quoque formula (11) demonstrari potest. Si enim fractiones (11) ad unum eundemque revocemus denominatorem, fiet:

$$\alpha_k = \frac{-1^k A_1 (A+A_1)^{k-1} + (2^k-2) A_1^2 (A+A_1)^{k-2} -}{1 \,.\, 2 \,.\, 3 \ldots (k-1)\, k\, (A+A_1)^k}$$

$$\frac{-(3^k-3.2^k+3) A_1^3 (A+A_1)^{k-3} + (4^k-4.3^k+6.2^k-4) A_1^4 (A+A_1)^{k-4} \text{ etc.}}{1 \,.\, 2 \,.\, 3 \ldots (k-1)\, k\, (A+A_1)^k}$$

Designantibus $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$, $\mathfrak{B}_4$ etc. singulos numeratoris terminos, et evolutis binomiis erit:

$$\mathfrak{B}_1 = -1^k A_1 A^{k-1} - \frac{(k-1)}{1} 1^k A_1^2 A^{k-2} - \frac{(k-1)(k-2)}{1 \,.\, 2} \,.\, 1^k A_1^3 A^{k-3} - \ldots .$$

$$\mathfrak{B}_2 = (2^k-2)\left(A_1^2 A^{k-2} + \frac{(k-2)}{1} A_1^3 A^{k-3} + \frac{(k-2)(k-3)}{1.2} A_1^4 A^{k-4} + \ldots .\right.$$

$$\mathfrak{B}_3 = -(3^k-3.2^k+3)\left(A_1^3 A^{k-3} + \frac{(k-3)}{1} A_1^4 A^{k-4} + \frac{(k-3)(k-4)}{1.2} A_1^5 A^{k-5} + \ldots\right.$$

$$\mathfrak{B}_4 = (4^k-4.3^k+6.2^k-4)\left(A_1^4 A^{k-4} + \frac{(k-4)}{1} A_1^5 A^{k-5} + \ldots .\right.$$

$$\mathfrak{B}_5 = -(5^k-5.4^k+10.3^k-10.2^k+5\left(A_1^5 A^{k-5} + \ldots .\right.$$

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Quodsi indicatam multiplicationem conficiamus atque expressiones inde profectas (ut sub 13) secundum potestates descendentes ipsius A ordinemus, prodit revera pro coefficientibus numericis:

$$\mathfrak{A}_1 = 1^k$$

$$\mathfrak{A}_2 = 2^k - (k+1)$$

$$\mathfrak{A}_3 = 3^k - (k+1) \,.\, 2^k + \frac{(k+1)\, k}{1 \,.\, 2}$$

$$\mathfrak{A}_4 = 4^k - (k+1) \,.\, 3^k + \frac{(k+1)k}{1 \,.\, 2} \,.\, 2^k - \frac{(k+1)\, k\, (k-1)}{1 \,.\, 2 \,.\, 3}$$

$$\mathfrak{A}_5 = 5^k - (k+1) \,.\, 4^k + \frac{(k+1)k}{1 \,.\, 2} \,.\, 3^k - \frac{(k+1)k(k-1)}{1 \,.\, 2 \,.\, 3} \,.\, 2^k + \ldots .$$

etc. etc. etc.

§. 5.

Facto $f(x) = e^x$ erit (1 §. 4):

$$\varphi(x) = Ae^x + A_1 e^{x+h} = e^x\left(A + A_1 e^h\right)$$

qui valor si in 12 §. 4 substituatur, evenit:

$$(A+A_1)\, e^x = e^x\left(A_1 e^h + A\right)\left(1 + \alpha_1 h + \alpha_2 h^2 + \alpha_3 h^3 + \alpha_4 h^4 + \ldots\right)$$

quare fit:

$$\frac{A+A_1}{A_1e^h+A}=1+\alpha_1h+\alpha_2h^2+\alpha_3h^3+\alpha_4h^4+\alpha^5h_5+\ldots.$$

Diviso numeratore et denominatore per A_1, et posito $\frac{A}{A_1}=b$ prodit:

$$\frac{b+1}{e^h+b}=1+\alpha_1h+\alpha_2h^2+\alpha_3h^3+\alpha_4h^4+\alpha_5h^5+\ldots.\text{ sive}$$

$$1)\ \frac{1-e^h}{e^h+b}=\alpha_1h+\alpha_2h^2+\alpha_3h^3+\alpha_4h^4+\alpha_5h^5+\ldots.\text{ et}$$

$$\alpha_k=\frac{1}{k!}\left(-\frac{\Phi_1^k}{b+1}-\frac{\Phi_2^k}{(b+1)^2}\ldots+(-1)^h\frac{\Phi_h^k}{(b+1)^h}+\ldots+(-1)^k\frac{\Phi_k^k}{(b+k)^k}\right)$$

(11. §. 5.)

Accepto denique $b=1$ habebimus:

$$2)\ \frac{1-e^h}{e^h+1}=\alpha_1h+\alpha_2h^2+\alpha_3h^3+\alpha_4h^4+\alpha_5h^5+\ldots.$$

$$\alpha_k=\frac{1}{k!}\left(-\frac{\Phi_1^k}{2}+\frac{\Phi_2^k}{2^2}+\ldots+(-1)^h\frac{\Phi_h^k}{2^h}+\ldots+(-1)^h\frac{\Phi_k^k}{2^k}\right)$$

$$3)\ \alpha_k=\frac{-2^{k-1}\Phi_1^k+2^{k-2}\Phi_2^k-\ldots+(-1)^h\,2^{k-h}\Phi_h^k+\ldots+(-1)^k\,\Phi_k^k}{1\,.\,2\,.\,3\,.\,4\,.\ldots(k-1)\,k\,.\,2^k}$$

At omne tum α, quod pari gaudet indice, evanescit; quod ut demonstretur, probare sufficit, pro eadem superiori assumtione coëfficientes numericos (in α_k 13,a §. 4), qui aeque longe ab ambobus remoti sint finibus, omnino aequos esse, ita ut fiat $\mathfrak{A}_1=\mathfrak{A}_k$, $\mathfrak{A}_2=\mathfrak{A}_{k-1}$, $\mathfrak{A}_3=\mathfrak{A}_{k-2}$ etc. etc. Quum enim pro $k=2n$, $(n=1, 2, 3, 4\ldots)$ coefficientes hi parem tunc numerum conficiant, atque alternis sint praediti signis, facile intelligitur, unum altero tolli, siquidem nulli alii cum iis valores conjuncti sint. Jam vero pro α_1, α_2, α_3, α_4 etc. seriei (1) valoribus formarum sub 13 §. 4 usi invenimus generaliter:

$$4)\ \alpha_k=-\frac{\mathfrak{A}_1b^{k-1}-\mathfrak{A}_2b^{k-2}+\mathfrak{A}_3b^{k-3}-\ldots+(-1)^k\mathfrak{A}_{k-1}b+(-1)^{k+1}\mathfrak{A}_k}{1.2.3.4\ldots(k-1)k(b+1)^k}$$

ubi literae eae aequationibus (secundum 3 §. 4) posito $\frac{A}{A_1}=b$ sequentibus:

$$5)\ 1+\alpha_1(b+1)=0$$

$$\frac{1}{1.2}+\alpha_1+\alpha_2(b+1)=0$$

$$\frac{1}{1.2.3}+\frac{\alpha_1}{1.2}+\alpha_2+\alpha_3(b+1)=0$$

$$\frac{1}{1.2.3.4}+\frac{\alpha_1}{1.2.3}+\frac{\alpha_2}{1.2}+\alpha_3+\alpha_4(b+1)=0$$

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determinantur.

Assumto pro b valore reciproco sive $b = \frac{A_1}{A}$ et designatis tum coefficientibus seriei (1) per $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \ldots . \mathfrak{a}_k \ldots$ ita ut pro b posito $\frac{1}{b}$ fiat:

$$6)\ \frac{1 - e^h}{e^h + \frac{1}{b}} = \mathfrak{a}_1 h + \mathfrak{a}_2 h^2 + \mathfrak{a}_3 h^3 + \mathfrak{a}_4 h^4 + \mathfrak{a}_5 h^5 + \ldots .$$

$$7)\ \mathfrak{a}_k = - \frac{\mathfrak{A}_1 \left(\frac{1}{b}\right)^{k-1} - \mathfrak{A}_2 \left(\frac{1}{b}\right)^{k-2} + \ldots + (-1)^k \mathfrak{A}_{k-1} \left(\frac{1}{b}\right) + (-1)^{k+1} \mathfrak{A}_k}{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots (k-1)\, k \left(\frac{1}{b} + 1\right)^k}$$

ad definiendos hos coefficientes simili modo praebentur aequationes (secundum 3 §. 4.)

$$b + \mathfrak{a}_1 (b+1) = 0$$

$$\frac{b}{1.2} + \mathfrak{a}_1 b + \mathfrak{a}_2 (b+1) = 0$$

$$\frac{b}{1.2.3} + \mathfrak{a}_1 \frac{b}{1.2} + \mathfrak{a}_2 b + \mathfrak{a}_3 (b+1) = 0$$

$$\frac{b}{1.2.3.4} + \mathfrak{a}_1 \frac{b}{1.2.3} + \mathfrak{a}_2 \frac{b}{1.2} + \mathfrak{a}_3 b + \mathfrak{a}_4 (b+1) = 0$$

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sive $8)\ 1 + \mathfrak{a}_1 \left(\frac{b+1}{b}\right) = 0$

$$\frac{1}{1.2} + \mathfrak{a}_1 + \mathfrak{a}_2 \left(\frac{b+1}{b}\right) = 0$$

$$\frac{1}{1.2.3} + \frac{\mathfrak{a}_1}{1.2} + \mathfrak{a}_2 + \mathfrak{a}_3 \left(\frac{b+1}{b}\right) = 0$$

$$\frac{1}{1.2.3.4} + \frac{\mathfrak{a}_1}{1.2.3} + \frac{\mathfrak{a}_2}{1.2} + \mathfrak{a}_3 + \mathfrak{a}_4 \left(\frac{b+1}{b}\right) = 0$$

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Ex his igitur aequationibus (5 et 8) dilucide apparet, α et $\mathfrak{a}$ iisdem praedita indicibus, quod ad coefficientes numericos, signa, et numerum terminorum attineat, eosdem esse valores, atque solum litera b eos differi, quae quum in omni insequenti aequatione (sub 8) cum summo unoquoque $\mathfrak{a}$ conjuncta inveniatur ut denominator, qui in dissoluta forma (7) in numeratorem adscendat, tum eo quoque discrimine remoto illi valores penitus aequi fient. Quamobrem $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ldots \alpha_k \ldots$ per b si multiplicentur, aut $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \ldots . \mathfrak{a}_k \ldots$ per b si dividantur earum congruentium literarum altera alteri plane par exsistet. Itaque $b\alpha_1 = \mathfrak{a}_1$, $b\alpha_2 = \mathfrak{a}_2$, $b\alpha_3 = \mathfrak{a}_3 \ldots b\alpha_k = \mathfrak{a}_k$ fiet, quare

$$b\,\frac{\mathfrak{A}_1 b^{k-1} - \mathfrak{A}_2 b^{k-2} + \mathfrak{A}_3 b^{k-3} - \ldots + (-1)^k \mathfrak{A}_{k-1}\, b + (-1)^{k+1}\mathfrak{A}_k}{1\,.\,2\,.\,3\,.\,4\ldots(k-1)\,k\,(b+1)^k} =$$

$$= \frac{\mathfrak{A}_1\left(\frac{1}{b}\right)^{k-1} - \mathfrak{A}_2\left(\frac{1}{b}\right)^{k-2} + \ldots + (-1)^k\mathfrak{A}_{k-1}\left(\frac{1}{b}\right) + (-1)^{k+1}\mathfrak{A}_k}{1\,.\,2\,.\,3\,.\,4\ldots(k-1)\,k\left(\frac{1}{b}+1\right)^k}$$

$$\frac{\mathfrak{A}_1 b^k - \mathfrak{A}_2 b^{k-1} + \mathfrak{A}_3 b^{k-2} - \ldots + (-1)^k\mathfrak{A}_{k-1} b^2 + (-1)^{k+1}\mathfrak{A}_k b}{1\,.\,2\,.\,3\,.\,4\ldots(k-1)\,k\,(b+1)^k} =$$

$$= \frac{\mathfrak{A}_1 b - \mathfrak{A}_2 b^2 + \mathfrak{A}_3 b^3 - \ldots + (-1)^k\mathfrak{A}_{k-1} b^{k-1} + (-1)^{k+1}\mathfrak{A}_k b^k}{1\,.\,2\,.\,3\,.\,4\ldots(k-1)\,k\,(b+1)^k}$$

$$\mathfrak{A}_1 b^k - \mathfrak{A}_2 b^{k-1} + \mathfrak{A}_3 b^{k-2} - \ldots + (-1)^k\mathfrak{A}_{k-1} b^2 + (-1)^{k+1}\mathfrak{A}_k b =$$
$$= \mathfrak{A}_1 b - \mathfrak{A}_2 b^2 + \mathfrak{A}_3 b^3 - \ldots + (-1)^k\mathfrak{A}_{k-1} b^{k-1} + (-1)^{k+1}\mathfrak{A}_k b^k$$

unde proficiscitur:

$$\mathfrak{A}_1 = (-1)^{k+1}\mathfrak{A}_k,\quad -\mathfrak{A}_2 = (-1)^k\mathfrak{A}_{k-1},\quad \mathfrak{A}_3 = (-1)^{k-1}\mathfrak{A}_{k-2} \text{ etc. etc.}$$

Facile ergo intelligitur pro $k = 2n$ posito $n = 1, 2, 3, 4 \ldots$ atque $b = 1$ omne α evanescere; quae conditio si in seriem (2) recipiatur, evenit:

$$\frac{1 - e^h}{e^h + 1} = \alpha_1 h + \alpha_3 h^3 + \alpha_5 h^5 + \ldots + \alpha_{2k-1} h^{2k-1} + \ldots$$

Posito nunc $h = 2vi$ (ubi $i = \sqrt{-1}$) et fractionis sinistrae partis numeratore et denominatore per e^{-vi} multiplicato, fiet:

$$\frac{e^{-vi} - e^{vi}}{e^{vi} + e^{-vi}} = -\frac{2i\sin v}{2\cos v} = 2i\,\alpha_1 . v - 2^3 i\,\alpha_3 . v^3 + 2^5 i\,\alpha_5 . v^5 - \ldots$$
$$\pm\, 2^{2k-1} i\,\alpha_{2k-1} . v^{2k-1} \ldots .$$

9) $-\operatorname{tag} v = 2\alpha_1 . v - 2^3\alpha_3 . v^3 + 2^5\alpha_5 . v^5 - \ldots \pm 2^{2k-1}\alpha_{2k-1} v^{2k-1} \ldots$

Quum autem constare quoque notum sit:

10) $-\operatorname{tag} v = -\frac{2^2(2^2-1)}{1\,.\,2} B^1 . v - \frac{2^4(2^4-1)}{1\,.\,2\,.\,3\,.\,4} B^2 . v^3 - \frac{2^6(2^6-1)}{1\,.\,2\,.\,3\ldots 6} B^3 . v^5 - \ldots$
$$- \frac{2^{2k}(2^{2k}-1)}{1\,.\,2\,.\,3\,..\,2k} B^k . v^{2k-1} - \ldots$$

ubi $B^1, B^2, B^3 \ldots B^k$ .. numeri sunt Bernoulliani, comparatis ergo his valoribus 9 et 10, prodit:

$$-2\alpha_1 = \frac{2^2(2^2-1)}{1\,.\,2} B^1$$

$$+2^3\alpha_3 = \frac{2^4(2^4-1)}{1\,.\,2\,.\,3\,.\,4} B^2$$

$$-2^5\alpha_5 = \frac{2^6(2^6-1)}{1\,.\,2\ldots 6} B^3$$

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$$\pm\, 2^{2k-1}\alpha_{2k-1} = \frac{2^{2k}(2^{2k}-1)}{1\,.\,2\,.\,3\,..\,2k} B^k, \pm \text{ pro } k \left(\begin{matrix}\text{pari}\\ \text{impari}\end{matrix}\right)$$

Quoniam vero est secundum 3:

$$\alpha_{2k-1} = \frac{-2^{2k-2}\Phi_1^{2k-1} + 2^{2k-3}\Phi_2^{2k-1} - 2^{2k-4}\Phi_3^{2k-1} + \ldots + 2\Phi_{2k-2}^{2k-1} - \Phi_{2k-1}^{2k-1}}{1\,.\,2\,.\,3\,.\,4\ldots(2k-1)\,.\,2^{2k-1}}$$

erit igitur:

$$\frac{2^{2k}(2^{2k}-1)}{1.2.3..2k}B^k = \pm \frac{-2^{2k-2}\Phi_1^{2k-1} + 2^{2k-3}\Phi_2^{2k-1} - \ldots + 2\Phi_{2k-2}^{2k-1} - \Phi_{2k-1}^{2k-1}}{1.2.3.4\ldots(2k-1)}$$

$$\pm \text{ pro k } \binom{\text{pari}}{\text{impari}}$$

quare valorem non dependentem k^{ti} numeri Bernoulliani nanciscimur:

$$11)\; B^k = \pm \frac{2k}{2^{2k}(2^{2k}-1)}\left(-2^{2k-2}\Phi_1^{2k-1} + 2^{2k-3}\Phi_2^{2k-1} - \ldots + 2\Phi_{2k-2}^{2k-1} - \Phi_{2k-1}^{2k-1}\right)$$

$$\pm \text{ pro k } \binom{\text{pari}}{\text{impari}}$$

Sin autem accipiatur $b = -1$ i. e. $A + A_1 = 0 = B$ (1. §. 3) erit secundum 3. §. 3.

$$f(x) = \frac{1}{B_1}\Sigma\varphi(x) = \frac{1}{A_1}\Sigma\varphi(x)$$

quare fiet secundum 7 §. 3.

$$12)\; \frac{A_1}{A_1}\Sigma\varphi(x) = \Sigma\varphi(x) = \frac{1}{h}\int\varphi(x)\,dx + \alpha_1\varphi(x) + \alpha_2 h\varphi'(x) +$$
$$+ \alpha_3 h^2\varphi''(x) + \ldots + C$$

Coëfficientes vero α_1, α_2, α_3, α_4 etc. aequationibus ex 20. §. 3 determinantur his:

$$\frac{\Phi_1^2}{1\,.\,2} + \alpha_1 = 0$$

$$\frac{\Phi_1^3}{1.2.3} + \alpha_1\frac{\Phi_1^2}{1\,.\,2} + \alpha_2 = 0$$

$$\frac{\Phi_1^4}{1.2.3.4} + \alpha_1\frac{\Phi_1^3}{1.2.3} + \alpha_2\frac{\Phi_1^2}{1\,.\,2} + \alpha_3 = 0$$

$$\frac{\Phi_1^5}{1.2.3.4.5} + \alpha_1\frac{\Phi_1^4}{1.2.3.4} + \alpha_2\frac{\Phi_1^3}{1.2.3} + \alpha_3\frac{\Phi_1^2}{1\,.\,2} + \alpha_4 = 0$$

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qui quoque substitutione $\varphi(x) = e^x$ in 12 facta simili modo ac supra definiri atque computari possunt. At multo facilius ad constituendos eos pervenimus, si aliam summae expressionem jam notam cum illa sub 12 conferimus, quippe quae pro omni functione ipsius x valeat necesse est. Itaque neminem latet esse:

$$\Sigma x^n = \frac{1}{h}\frac{x^{n+1}}{n+1} - \frac{1}{2}x^n + B^1\frac{n}{1}.\frac{h}{2}x^{n-1} - B^2\frac{n(n-1)(n-2)}{1.2.3}.\frac{h^3}{4}.x^{n-3} +$$

$$+B^3 \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \cdot \frac{h^5}{6} x^{n-5} - \ldots$$

$$\pm B^k \frac{n(n-1)\ldots(n-2k+2)}{1.2.3\ldots(2k-1)} \cdot \frac{h^{2k-1}}{2k} x^{n-2k+1} \mp \ldots$$

ubi B^1, B^2, B^3 . . B^k numeri sunt Bernoulliani; et posito in 12 $\varphi(x) = x^n$ prodit:

$$\Sigma x^n = \frac{1}{h} \frac{x^{n+1}}{n+1} + \alpha_1 x^n + n\alpha_2 h x^{n-1} + n(n-1)\alpha_3 h^2 x^{n-2} + \ldots$$

$$+ n(n-1)(n-2)\ldots(n-2k+2)\alpha_{2k} h^{2k-1} x^{n-2k+1} + \ldots$$

Quodsi congruentes comparantur potestates ipsius x, prodit:

$$\alpha_1 = -\frac{1}{2}$$

$$n\alpha_2 h = +B^1 \frac{n}{1} \cdot \frac{h}{2}$$

$$n(n-1)\alpha_3 h^2 = 0$$

$$n(n-1)(n-2)\alpha_4 h^3 = -B^2 \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{h^3}{4}$$

$$n(n-1)(n-2)(n-3)\alpha_5 h^4 = 0$$

$$n(n-1)(n-2)(n-3)(n-4)\alpha_6 h^5 = +B^3 \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \cdot \frac{h^5}{6}$$

et omnino

$$n(n-1)(n-2)\ldots(n-2k+2)\alpha_k h^{2k-1} = \pm B^k \frac{n(n-1)(n-2)\ldots(n-2k+2)}{1.2.3\ldots(2k-1)} \cdot \frac{h^{2k-1}}{2k}$$

$$\pm \text{ pro } k \begin{pmatrix}\text{impari}\\ \text{pari}\end{pmatrix}$$

quare fit:

$$\alpha_1 = -\frac{1}{2}$$

$$\alpha_2 = +\frac{B^1}{1.2}$$

$$\alpha_3 = 0$$

$$\alpha_4 = -\frac{B^2}{1.2.3.4}$$

$$\alpha_5 = 0$$

$$\alpha_6 = +\frac{B^3}{1.2.3.4.5.6}$$

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$$\alpha_{2k} = \pm \frac{B^k}{1.2.3\ldots 2k} \pm \text{ pro } k \begin{pmatrix}\text{imperi}\\ \text{pari}\end{pmatrix}$$

Quum autem numeri Bernoulliani pro omni k ex 11 constitui possint, substitutis in aeq 12 pro α_1, α_2, α_3, α_4 etc. etc. valoribus, series inde proficiscens:

$$13)\ \Sigma\varphi(x)=\frac{1}{h}\int\varphi(x)dx-\frac{1}{2}\varphi(x)+\frac{B^1}{1\,.\,2}h\varphi'(x)-\frac{B^2}{1.2.3.4}h^3\varphi'''(x)+\ldots$$
$$\pm\frac{B^k}{1\,.\,2\,.\,3\ldots2k}h^{2k-1}\varphi^{2k-1}(x)\ldots$$

si detur $\varphi(x)$, adjiciaturque constans C, quae facto $x = 0$ aut pro x alio quodam substituto valore determinatur, plane nota erit.

§. 6.

Quoniam ex 11 §. 5 prodit: $B^1=\frac{1}{6}, B^2=\frac{1}{30}, B^3=\frac{1}{42}, B^4=\frac{1}{30}$ $B^5=\frac{5}{66}, B^6=\frac{691}{2730}, B^7=\frac{7}{6}, B^8=\frac{3617}{510}, B^9=\frac{43867}{798}$ etc. posito $h=1$ formula 13, §. 5 abibit in hanc:

$$1)\ \Sigma\varphi(x)=\int\varphi(x)dx-\frac{1}{2}\varphi(x)+\frac{1}{12}\varphi'(x)-\frac{1}{720}\varphi'''(x)+$$
$$+\frac{1}{30240}\varphi'''''(x)\ldots$$

quae in summandis seriebus numericis amplissimum habere potest usum. Quum enim impares solum differentiales quotientes exstent, exponentes quantitatis variabilis differentiando celerrime decrescunt, itaque ad constantem demum perveniatur terminum necesse est, quem ceteri sequentes ubi evanescunt, summa $\Sigma\varphi(x)$ vel terminus summatorius per seriem finitam atque conclusam exprimitur. Ponatur e. g. indicibus localibus:

1, 2, 3, 4 . . . . . x respondere seriem

$1 + 3 + 5 + 7 \ldots + (2x-1)$ cujus summa sit $= \Sigma\varphi(x)$ tum habebimus $\Delta\Sigma\varphi(x) = \varphi(x) = 2x+1$, qui valor si in formula (1) substituatur, evadit:

$$\Sigma\varphi(x) = x^2 + C$$

Quum vero pro $x = 0$ summa evanescat, constans $C=0$ quoque fiet, atque erit:

$$\Sigma\varphi(x) = x^2$$

2. Generaliter sit series data, cujus terminus primus $= a$, atque differentia prima constans $= \Delta a$ est; quum enim terminus generalis hujus snriei, cujus summam $\Sigma\varphi(x)$ ponimus, $= a + \Delta a(x-1)$ fiat, erit tum:

$$\varphi(x) = a + \Delta ax, \text{ nanciscimur ergo:}$$

$$\int\varphi(x)dx = \frac{\Delta a}{2}x^2 + ax + c$$
$$-\frac{1}{2}\varphi(x) = \ldots -\frac{1}{2}\Delta ax - \frac{1}{2}a$$
$$+\frac{1}{12}\varphi'(x) = \quad \ldots \quad +\frac{\Delta a}{12}$$
$$-\frac{1}{720}\varphi'''(x) = 0 \text{ quare fit:}$$

$\Sigma\,\varphi\,(x) = \frac{\Delta a}{2}\,x^2 + (a - \frac{\Delta a}{2})\,x + C$, quae constans C ita comparata esse debet, ut posito $x = 0$ summa quoque $\Sigma\,\varphi\,(x)$ evanescat.

Parum etiam difficultatis exhibetur, si termini seriei propositae ad secundam, tertiam etc. etc. constantem differentiam perducant, quippe enim generales tum termini fient: $a + \Delta a(x-1) + \Delta^2 a \frac{(x-1)(x-2)}{1\,.\,2}$; $a + \Delta a(x-1) + \Delta^2 a\,.\,\frac{(x-1)\,(x-2)}{1\,.\,2} + \Delta^3\,a\,.\,\frac{(x-1)\,(x-2)\,(x-3)}{1\,.\,2\,.\,3}$ etc. etc. facillime $\varphi\,(x)$ indidemque $\Sigma\,\varphi\,(x)$ determinari potest.

3) Quodsi data est series, cujus termini fractiones sunt, ita ut quoque e. g. $\varphi\,(x) = \frac{1}{x}$ fiat, constans C commemorato jam modo determinari nequit; quum enim sit

$$\int \varphi(x)\,dx = l(x)$$
$$-\frac{1}{2}\,\varphi\,(x) = -\frac{1}{2x}$$
$$+\frac{1}{12}\varphi'(x) = -\frac{1}{12x^2}$$
$$-\frac{1}{720}\,\varphi'''(x) = +\frac{6}{720x^4}$$

prodit: $\Sigma\varphi\,(x) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + .. + \frac{1}{x} = l(x) - \frac{1}{2x} - $

$$-\frac{1}{12x^2} + \frac{6}{720x^4} \ldots + c$$

ubi termini singuli facto $x = 0$ infinite magni evadunt, constansque addenda definiri non potest. Ponatur autem $x = 1$, quia tum est $\Sigma\,\varphi\,(x) = 1$, ob causam summandi unum solum terminum, qui est $= 1$, erit:

$$1 = -\frac{1}{2} - \frac{1}{12} + \frac{1}{120} + \ldots + c$$

$c = \frac{3}{2} + \frac{1}{12} - \frac{1}{120} - \ldots$ quamobrem

$\Sigma\varphi\,(x) = l\,(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} + \ldots + \frac{3}{2} + \frac{1}{12} - \frac{1}{120} - \ldots$ aut

$\Sigma\varphi\,(x) = l\,(x) - \frac{1}{2x} - \frac{B^1}{2x^2} + \frac{B^2}{4x^4} - \frac{B^3}{6x^6} \ldots + \frac{3}{2} + \frac{B^1}{2} - \frac{B^2}{4} + \ldots$

si recipiantur numeri Bernoulliani, qui quum semiconvergentem praebeant seriem, summa nonnisi per appropinquationem inveniri potest. Perspicuum quoque est, quo major pro x sumatur valor, eo accuratius determinari constantem.

4. Praeterea formula illa (1) adhiberi potest summandis seriebus duplibus i. e. seriebus, quarum termini producta sunt duorum aliorum, quorum alter alteram sequitur legem. Sit e. g. series:

$$\overset{1}{1^2} + \overset{2}{3 \cdot 2^2} + \overset{3}{5 \cdot 3^2} + \overset{4}{7 \cdot 4^2} + \ldots + \overset{x}{(2x-1)x^2}$$ cujus summa $= \Sigma\varphi(x)$, quia est $\varphi(x) = (2x+1)(x+1)^2$, summa quoque, quum sit:

$$\int \varphi(x)\,dx = \frac{2}{4}x^4 + \frac{5}{3}x^3 + 2x^2 + x + c$$
$$-\frac{1}{2}\varphi(x) = \ldots - x^3 - \frac{5}{2}x^2 - 2x - \frac{1}{2}$$
$$+\frac{1}{12}\varphi'(x) = \ldots\ldots + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{1}{3}$$
$$-\frac{1}{720}\varphi'''(x) = \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{60}$$

$\Sigma\varphi(x) = \frac{1}{2}x^4 + \frac{2}{3}x^3 - \frac{1}{6}x + C$ invenitur ubi, constans $C = 0$ fit, quia facto $x = 0$ ob summandum nullum terminum $\Sigma\varphi(x) = 0$ quoque erit.

Detur generaliter series:

$$\Sigma\varphi(x) = \overset{1}{a} + \overset{2}{2^k (a + \Delta a)} + \overset{3}{3^k (a + 2\Delta a)} + \overset{4}{4^k (a + 3\Delta a)} + \ldots + \overset{x}{x^k (a + (x-1)\Delta a)}$$

erit $\varphi(x) = (x+1)^k (a + x\Delta a)$; quum vero sit

$$\int\varphi(x)\,dx = \int \left\{ a(x+1)^k + \Delta a\, x (x+1)^k \right\} dx = \int a (x+1)^k dx + + \int \Delta a\, x (x+1)^k dx$$

atque:

$$\int a(x+1)^k dx = a\frac{x^{k+2}}{k+1} + \frac{k}{1} a \frac{x^k}{k} + \frac{k(k-1)}{1.2} a \cdot \frac{x^{k-1}}{k-1} + \ldots$$

$$\int \Delta a\, x (x+1)^k dx = \Delta a \frac{x^{k+1}}{k+2} + \frac{k}{1} \cdot \Delta a \frac{x^{k+1}}{k+1} + \frac{k(k-1)}{1.2} \cdot \Delta a \frac{x^k}{k} + \ldots$$

fiet:

$$\int\varphi(x)\,dx = \Delta a \frac{x^{k+2}}{k+2} + \left(a + k\Delta a\right)\frac{x^{k+1}}{k+1} + \left(ka + \frac{k(k-1)}{1.2}.\Delta a\right)\frac{x^k}{k} + $$
$$+ \left(\frac{k(k-1)}{1.2} a + \frac{k(k-1)(k-2).\Delta a}{1.2.3}\right)\frac{x^{k-1}}{k-1} + \left(\frac{k(k-1)(k-2)}{1.2.3} . a + \right.$$
$$\left. + \frac{k(k-1)(k-2)(k-3)}{1.2.3.4}\Delta a\right)\frac{x^{k-2}}{k-2} + \ldots$$

Erit quoque:

$$-\frac{1}{2}\varphi(x) = -\frac{1}{2}\Big((x+1)^k\ (a + x\Delta a)\Big)$$
$$+\frac{B^1}{1.2}\varphi'(x) = +\frac{1}{12}\Big(k\ (x+1)^{k-1}\ (a + x\Delta a) + (x+1)^k \Delta a\Big)$$

$$-\frac{B^2}{1.2.3.4}\varphi'''(x) = -\frac{1}{720}\left(k(k-1)(k-2)(x+1)^{k-3}.(a+x\Delta a)+\right.$$
$$\left.+\,3k(k-1)(x+1)^{k-2}\Delta a\right)$$

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quare summa $\Sigma\,\varphi\,(x)$ facillime constitui potest. Quodsi superiorem sub N. 4 nancisci volumus seriem, $a = 1$, $\Delta a = 2$, $k = 2$ ponatur necesse est.

Neque rem multum habere difficultatis, si partes singulae terminorum seriei duplicis unam eandemque sequantur legem, plane liquet.

§. 7.

Sin autem serierum duplicium una ejusmodi est, ut exponentes terminorum ipsius functiones sint indicum localium, quare terminus generalis quantitas exponentialis e. g. $= p^x$ fiat, formula 1. §. 6 prosperis solum in casibus summandis seriebus adhiberi potest, tum praesertim, quum ex integrali termini primi atque differentiatione ceterorum profectae series logarithmorum convergant. Sed alia derivari potest formula, quae quum et integrali, quia quoque $f(x) = \Sigma\varphi(x)$ pollet, quamquam $B = 0$ §. 5 non fit, et quotientibus differentialibus quantitatis exponentialis egeat, generi ei serierum expeditissime accommodatur. Posito enim in 12 §. 4 e^x loco $\varphi(x)$ prodit:

$$(A+A_1)\; f(x) = e^x \left(1 + \alpha_1 h + \alpha_2 h^2 + \alpha_3 h^3 + \alpha_4 h^4 + \ldots\right)$$

quum vero series parenthesi inclusa ex evolutione fractionis $\dfrac{A + A_1}{A_1 e^h + A}$ proficiscatur §. 5, nanciscimur:

$$(A+A_1)\; f(x) = e^x \,.\, \frac{A+A_1}{A_1 e^h + A}, \text{ quare fit:}$$

$$f(x) = \frac{e^x}{A_1 e^h + A}$$

Quodsi $A_1 = p^{h-x}$ atque $A = -p^{-x}$ sit (i. e. B et B_1 functiones ipsius x), prodit:

$$f(x) = \frac{p^x e^x}{p^h e^h - 1} = \Sigma\, p^x e^x = \Sigma p^x \varphi\,(x)$$

Qua substitutione in 12 §. 4 quoque confecta erit:

$$\alpha_1 = -\frac{p^h}{p^h - 1}$$
$$\alpha_2 = \frac{p^{2h} + h^h}{1.2\,(p^h-1)^2}$$

$$\alpha_3 = -\frac{p^{3h} + 4p^{2h} + p^h}{1.2.3\,(p^h - 1)^3}$$

$$\alpha_4 = \frac{p^{4h} + 11p^{3h} + 11p^{2h} + p}{1.2.3.4\,(p^h - 4)^4}$$

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$$\alpha_k = (-1)^k \frac{A_1 p^{kh} + \mathfrak{A}_2 p^{(k-1)h} + \mathfrak{A}_3 p^{(k-2)h} + \ldots + \mathfrak{A}_{k-1} p^{2h} + \mathfrak{A}_k p^h}{1.2.3.4\ldots k\,(p^h - 1)^k}$$

ubi $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ etc. etc. etc. coefficientes numerici jamjam sunt noti.

Si demum valores hi in aeq. 11. §. 4 substituantur, atque ponatur $h = 1$ et generaliter $p^x = \psi(x)$, dum meminerimus $\psi(x)$ esse quantitatem exponentialem, cujus basis est p, habebimus:

$$\left(\frac{p-1}{\psi(x)}\right) \Sigma\psi(x)\varphi(x) = \varphi(x) - \frac{p}{p-1}\varphi'(x) + \frac{p^2+p}{1.2\,(p-1)^2}\varphi''(x) - \\ - \frac{p^3+4p^2+p}{1.2.3\,(p-1)^3}\varphi'''(x) + \ldots$$

$$1)\ \Sigma\psi(x)\varphi(x) = \frac{\psi(x)}{p-1}\left(\varphi(x) - \frac{p}{p-1}\varphi'(x) + \frac{p^2+p}{1.2\,(p-1)^2}\varphi''(x) - \\ - \frac{p^3+4p^2+p}{1.2.3\,(p-1)^3}\varphi'''(x) + \ldots\right) + C$$

Sit e. g. series:

$$\Sigma\psi(x)\varphi(x) = \overset{1}{p} + \overset{2}{p^2 . 2^k} + \overset{3}{p^3 . 3^k} + \overset{4}{p^4 . 4^k} + \ldots + \overset{x}{p^x . x^k}$$

et quum fiat: $\psi(x)\varphi(x) = p^{x+1}(x+1)^k$, ponatur oportet: $\psi(x) = p^{x+1}$, $\varphi(x) = (x+1)^k$, qui valores si in (1) collocentur prodit:

$$2)\ \Sigma\psi(x)\varphi(x) = \frac{p^{x+1}}{p-1}\Big((x+1)^k - k\frac{p}{p-1}(x-1)^{k-1} + \\ + \frac{k(k-1)}{1.2}\frac{p^2+p}{(p-1)^2}(x+1)^{k-2} - \frac{k(k-1)(k-2)}{1.2.3}\frac{p^3+4p^2+p}{(p-1)^3}(x+1)^{k-3} + \\ + \ldots\Big) + C$$

Constans arbitraria C simili modo ac supra demonstravimus, definitur; quum enim facto $x = 0$, $\Sigma\psi(x)\varphi(x) = 0$ fiat, quia nulli, qui summentur termini praesto sunt, evenit: $0 = A + C$ quare $C = -A$, ubi A quantitas est semper constans et definita.

Ut jam ea, quae modo ostendimus, probentur, ponatur $k = 2$, quare fit:

$$\Sigma\psi(x)\varphi(x) = \frac{p^{x+1}}{p-1}\left((x+1)^2 - 2\frac{p}{p-1}(x+1) + \frac{p^2+p}{(p-1)^2}\right) + C;$$

facto vero $x = 0$ aequatio, qua constans arbitraria definitur, procedit:

$$0 = \frac{p}{p-1}\left(1 - \frac{2p}{p-1} + \frac{p^2+p}{(p-1)^2}\right) + C, \text{ itaque}$$

*

$$C = -\frac{p^2+p}{(p-1)^3} \text{ et summa completa:}$$

$$3) \Sigma\psi(x)\varphi(x) = \frac{p^{x+1}}{p-1}\left((x+1)^2 - 2 \cdot \frac{p}{p-1}(x+1) + \frac{p^2+p}{(p-1)^2}\right) - \\ - \frac{p^2+p}{(p-1)^3} \text{ erit.}$$

Posito nunc e. g. $p = 4$ et $x = 6$ series summanda erit:

$$\Sigma\psi(x)\varphi(x) = \overset{1}{4} + \overset{2}{4^2 . 2^2} + \overset{3}{4^3 . 3^2} + \overset{4}{4^4 . 4^2} + \overset{5}{4^5 . 5^2} + \overset{6}{4^6 . 6^2} = 177796$$

et substitutis valoribus his in 3, prodit:

$$\Sigma\psi(x)\varphi(x) = \frac{4^7}{3}\left(7^2 - 2 . \frac{4}{3} . 7 + \frac{4^2+4}{3^2}\right) - \frac{4^2+4}{3^3} = \frac{4^7}{3} . \frac{293}{9} - \\ - \frac{20}{27} = 177796$$

Posito $k = 0$, quum sit $\varphi(x) = 1$ obtinebimus:

$$\Sigma\psi(x) = \frac{p^{x+1}}{p-1} + C \text{ et facto } x = 0 \text{ erit } C = -\frac{p}{p-1}, \text{quare habetur:}$$

$$\Sigma\psi(x) = \frac{p^{k+1}}{p-1} - \frac{p}{p-1} = \frac{p(p^x-1)}{p-1} \text{ nota summa progressionis:}$$

$$p + p^2 + p^3 + p^4 + \ldots + p^x$$

Numerum terminorum, quibus summa seriei omnino conflatur, e gradu ipsius $\varphi(x)$ pendere, facile intelligitur, quod quum secundum quamlibet legem conformari possit, excepta ea, qua quidem exponentes terminorum seriei functiones fiunt indicum localium, permultas, quae summentur, inveniri posse series apparet. Et si $p < 1$ ponatur, serierum fractarum quoque summae nonnullae inveniuntur.

Detur etiam series:

$\Sigma\psi(x)\varphi(x) = e^{vi} + 2e^{2vi} + 3e^{3vi} + 4e^{4vi} + \ldots + xe^{xvi}$, ubi e sit basis logarithmi naturalis et $i = \sqrt{-1}$ quamobrem ut ista summetur $\psi(x) = e^{(x+1)vi}$ et $\varphi(x) = x + 1$ ponatur necesse est, quibus valoribus in (1) substitutis, procedit:

$$\Sigma\psi(x)\varphi(x) = \frac{e^{(x+1)vi}}{e^{vi}-1}\left((x+1) - \frac{e^{vi}}{e^{vi}-1}\right) + C \text{ et determinandae con-}$$

stantis causa facto $x = 0$, quia fit $\Sigma\psi(x)\varphi(x) = 0$, evenit:

$$0 = \frac{e^{vi}}{e^{vi}-1}\left(1 - \frac{e^{vi}}{e^{vi}-1}\right) + C$$

$$C = \frac{e^{vi}}{(e^{vi}-1)^2} \text{ erit ergo summa completa:}$$

$$\Sigma\psi(x)\varphi(x) = \frac{e^{(x+1)vi}}{e^{vi}-1}\left((x+1) - \frac{e^{vi}}{e^{vi}-1}\right) + \frac{e^{vi}}{(e^{vi}-1)^2} = \\ = \frac{xe^{(x+2)vi} - (x+1)e^{(x+1)vi} + e^{vi}}{(e^{vi}-1)^2}$$

$$\frac{xe^{(x+2)vi} - (x+1)e^{(x+1)vi} + e^{vi}}{(e^{vi}-1)^2} = e^{vi} + 2e^{2vi} + 3e^{3vi} + \ldots + xe^{xvi};$$

fiet quoque:

$$\frac{xe^{-(x+2)vi}-(x+1)e^{-(x+1)vi}+e^{-vi}}{(e^{-vi}-1)^2}=e^{-vi}+2e^{-2vi}+3e^{-3vi}+\ldots+xe^{-xvi}$$

Additis his expressionibus et fractionibus partis sinistrae ad unum eundemque revocatis denominatorem obtinebimus:

$$\frac{x\left(e^{(x+2)}+e^{-(x+2)vi}\right)-(3x+1)\left(e^{(x+1)vi}+e^{-(x+1)vi}\right)+}{(e^{2vi}+e^{-2vi})-4(e^{vi}+e^{-vi})+6}$$

$$\frac{+(3x+2)(e^{xvi}+e^{-xvi})-(x+1)(e^{(x-1)vi}+e^{-(x-1)vi})+2(e^{vi}+e^{-vi})-4}{(e^{2vi}+e^{-2vi})-4(e^{vi}+e^{-vi})+6}=$$

$$=(e^{vi}+e^{-vi})+2(e^{2vi}+e^{-2vi})+3(e^{3vi}+e^{-3vi}+\ldots+x(e^{xvi}+e^{-xvi})$$

Iisdemque valoribus altero ab altero subtractis prodit:

$$\frac{x\left(e^{(x+2)vi}-e^{-(x+2)vi}\right)-(3x+1)\,e^{(x+1)vi}-e^{-(x+1)vi})+}{(e^{2vi}+e^{-2vi})-4(e^{vi}+e^{-vi})+6}$$

$$\frac{+(3x+2)(e^{xvi}-e^{-xvi})-(x+1)(e^{(x-1)vi}-e^{-(x-1)vi})}{(e^{2vi}+e^{-2vi})-4(e^{vi}-e^{-vi})+6}=$$

$$=(e^{vi}-e^{-vi})+2(e^{2vi}-e^{-2vi})+3(e^{3vi}-e^{-3vi})+\ldots+x(e^{xvi}-e^{-xvi})$$

Quum vero sit $\frac{e^{vi}+e^{-vi}}{2}=\cos v$, $\frac{e^{vi}-e^{-vi}}{2\sqrt{-1}}=\sin v$, et omnino

$e^{\pm xvi}=(\cos v\pm\sqrt{-1}\,\sin v)^x=\cos xv\pm\sqrt{-1}\,\sin xv$ erit denique:

$$4)\ \frac{x\cos(x+2)v-(3x+1)\cos(x+1)v+(3x+2)\cos xv-(x+1)\cos(x-1)v+}{2\cos 2v-2.4\cos v+6}$$

$$\frac{+2\cos v-2}{2\cos 2v-2.4\cos v+6}=$$

$$=\cos v+2\cos 2v+3\cos 3v+4\cos 4v+\ldots+x\cos xv$$

$$5)\ \frac{x\sin(x+2)v-(3x+1)\sin(x+1)v+(3x+2)\sin xv-}{2\cos 2v-2.4\cos v+6}$$

$$\frac{-(x+1)\sin(x-1)v}{2\cos 2v-2.4\cos v+6}=$$

$$=\sin v+2\sin 2v+3\sin 3v+4\sin 4v+\ldots+x\sin xv.$$

Accepto nunc in formula 4ta $v=\pi$, quum sit generaliter $\cos x\pi=\pm 1$ prout x par aut impar est, fiet

$$\frac{x\cos(x+2)\pi-(3x+1)\cos(x+1)\pi+(3x+2)\cos x\pi-}{2\cos 2\pi-2.4\cos\pi+6}$$

$$\frac{-(x+1)\cos(x-1)\pi+2\cos\pi-2}{2\cos 2\pi-2.4\cos\pi+6}=$$

$$=-1+2-3+4\ldots\pm x,\ \pm \text{ pro } x\begin{pmatrix}\text{pari}\\ \text{impari}\end{pmatrix}$$

Quoniam autem pro x pari est $\cos(x+2)\pi=+1$, $\cos(x+1)\pi=-1$ $\cos x\pi=+1$, $\cos(x-1)\pi=-1$, $\cos 2\pi=+1$, atque $\cos\pi=-1$, invenitur ergo:

$$\frac{8x}{16}=\frac{x}{2}=-1+2-3+4-5+6-\ldots+x \text{ aut}$$

$$-\frac{x}{2} = 1 - 2 + 3 - 4 + 5 - 6 + \ldots - x \text{ pro x pari.}$$

Quum vero pro x impari fiat $\cos(x+2)\pi = -1$, $\cos(x+1)\pi = +1$, $\cos x\pi = -1$, $\cos(x-1)\pi = +1$ prodit:

$$-\frac{8x+8}{16} = -\frac{x+1}{2} = -1 + 2 - 3 + 4 - 5 + 6 - \ldots - x \text{ aut}$$

$$\frac{x+1}{2} = 1 - 2 + 3 - 4 + 5 - 6 + \ldots + x \text{ pro x impari.}$$

Posito autem $v = \frac{\pi}{2}$ tum apparet pro x impari esse $\cos x \frac{\pi}{2} = 0$, pro x vero pari sive $x = 2y$ esse $\cos 2y \frac{\pi}{2} = \pm 1$ prout y par aut impar est. Quodsi in eadem formula substituatur 2y pro x sumto $y = 1, 2, 3, 4 \ldots$ nanciscimur:

$$\frac{2y\cos(2y+2)\frac{\pi}{2} - (6y+1)\cos(2y+1)\frac{\pi}{2} + (6y+2)\cos 2y\frac{\pi}{2} -}{2\cos 2\frac{\pi}{2} - 2 \,.\, 4\cos\frac{\pi}{2} + 6}$$

$$\frac{-(2y+1)\cos(2y-1)\frac{\pi}{2} + 2\cos\frac{\pi}{2} - 2}{2\cos 2\frac{\pi}{2} - 2 \,.\, 4\cos\frac{\pi}{2} + 6} =$$

$$= -\overset{1}{2} + \overset{2}{4} - \overset{3}{6} + \overset{4}{8} - \overset{5}{10} + \ldots \pm \overset{y}{2y},\ \pm \text{ pro y } \begin{pmatrix}\text{pari}\\ \text{impari}\end{pmatrix}$$

Quum vero pro y pari sit $\cos(2y+2)\frac{\pi}{2} = -1$, $\cos(2y+1)\frac{\pi}{2} = 0$, $\cos 2y\frac{\pi}{2} = +1$, $\cos(2y-1)\frac{\pi}{2} = 0$ prodit:

$$\frac{-2y+6y+2-2}{-2+6} = y = -\overset{1}{2} + \overset{2}{4} - \overset{3}{6} + \overset{4}{8} - \overset{5}{10} + \ldots + \overset{y}{2y} \text{ sive}$$

$$-y = 2 - 4 + 6 - 8 + 10 - \ldots - 2y \text{ pro y pari.}$$

Quoniam autem pro y impari est $\cos(2y+2)\frac{\pi}{2} = +1$, $\cos(2y+1)\frac{\pi}{2} = 0$, $\cos 2y\frac{\pi}{2} = -1$, $\cos(2y-1)\frac{\pi}{2} = 0$, evenit:

$$\frac{2y-6y-2-2}{-2+6} = -(y+1) = -\overset{1}{2} + \overset{2}{4} - \overset{3}{6} + \overset{4}{8} - \overset{5}{10} + \ldots - \overset{y}{2y} \text{ aut}$$

$$y + 1 = 2 - 4 + 6 - 8 + 10 + \ldots + 2y \text{ pro y impari.}$$

Quodsi in formula 5^{ta} ponatur $v = \frac{\pi}{2}$, erit generaliter $\sin x \frac{\pi}{2} = 0$

aut ± 1, quare omisso casu primo et pro x substituto $2y+1$, ($y=0,1,2,3,4\ldots$) fiet $\sin(2y+1)\frac{\pi}{2}=\pm 1$ prout y est numerus par aut impar. Itaque evadit:

$$\frac{(2y+1)\sin(2y+3)\frac{\pi}{2}-(6y+4)\sin(2y+2)\frac{\pi}{2}+(6y+5)\sin(2y+1)\frac{\pi}{2}-}{2\cos 2\frac{\pi}{2}-2\,.\,4\cos\frac{\pi}{2}+6}$$

$$\frac{-(2y+2)\sin 2y\frac{\pi}{2}}{2\cos 2\frac{\pi}{2}-2\,.\,4\cos\frac{\pi}{2}+6}=$$

$$=\overset{0}{1}-\overset{1}{3}+\overset{2}{5}-\overset{3}{7}+\overset{4}{9}-\overset{5}{11}+\overset{6}{13}-\ldots\pm\overset{y}{(2y+1)}$$

$$\pm \text{ pro } y\begin{pmatrix}\text{pari}\\ \text{impari}\end{pmatrix}$$

Quum vero pro y pari sit $\sin(2y+3)\frac{\pi}{2}=-1$, $\sin(2y+2)\frac{\pi}{2}=0$, $\sin(2y+1)\frac{\pi}{2}=+1$, $\sin 2y\frac{\pi}{2}=0$ habebimus:

$$\frac{-(2y+1)+(6y+5)}{-2+6}=(y+1)=\overset{0}{1}-\overset{1}{3}+\overset{2}{5}-\overset{3}{7}+\overset{4}{9}-\overset{5}{11}+$$

$$\ldots+\overset{y}{(2y+1)} \text{ pro } y \text{ pari.}$$

Quia pro y impari est $\sin(2y+3)\frac{\pi}{2}=+1$, $\sin(2y+2)\frac{\pi}{2}=0$, $\sin(2y+1)\frac{\pi}{2}=-1$, $\sin 2y\frac{\pi}{2}=0$, obtinebimus:

$$\frac{(2y+1)-(6y+5)}{-2+6}=-(y+1)=\overset{0}{1}-\overset{1}{3}+\overset{2}{5}-\overset{3}{7}+\overset{4}{9}-\overset{5}{11}+$$

$$\ldots-\overset{y}{(2y+1)} \text{ pro } y \text{ impari.}$$

Itaque ad aliud serierum genus pervenimus, quod quum e pluribus iterum speciebus constare possit, prout altera series cujus differentiam per $\varphi(x)$ denotavimus, comparata est, multas rursus impetramus alias, quae aut secundum sinus et cosinus multiplicium angulorum procedunt, aut sunt plane numericae, quarum termini alternis signis praditi se excipiunt.

Postremum hanc solum considerare liceat seriem:

$\Sigma\psi(x)\varphi(x)=\sin v+2^2\sin v^2+3^2\sin v^3,+4^2\sin v^4+\ldots+x^2\sin v^x$ quae ut summetur, $p=\sin v$, $\psi(x)=\sin v^{x+1}$, $\varphi(x)=(x+1)^2$ in formula 3 ponatur necesse est, quare summa fit:

$$\Sigma\psi(x)\varphi(x)=\frac{\sin v^{x+1}}{\sin v-1}\left((x+1)^2-2\frac{\sin v}{\sin v-1}(x+1)+\frac{\sin v^2+\sin v}{(\sin v-1)^2}\right)-$$

$$-\frac{\sin v^2+\sin v}{(\sin v-1)^3}$$

$$\Sigma\psi(x)\varphi(x)=\frac{x^2\sin v^{x+3}-(2x^2+2x-1)\sin v^{x+2}+}{(\sin v-1)^3}$$

$$\frac{+(x+1)^2\sin v^{x+1}-\sin v^2-\sin v}{(\sin v-1)^3}=$$

$$=\sin v+2^2\sin v^2+3^2\sin v^3+4^2\sin v^4+\ldots+x^2\sin v^x$$

Summam hanc veram esse singulis pro x valoribus sumtis comprobare possumus. Quamobrem posito $x=1$ erit:

$$\frac{\sin v^4-3\sin v^3+4\sin v^2-\sin v^2-\sin v}{(\sin v-1)^3}=\sin v$$

$$\frac{\sin v(\sin v^3-3\sin v^2+3\sin v-1)}{(\sin v-1)^3}=\sin v$$

$\sin v=\sin v$; posito $x=2$ procedit:

$$\frac{4\sin v^5-11\sin v^4+9\sin v^3-\sin v^2-\sin v}{(\sin v-1)^3}=\sin v+2^2\sin v^2;$$

quodsi ad numeratorem partis sinistrae addamus et subtrahamus $\sin v^4$, $8\sin v^3$, $3\sin v^2$ prodit:

$$\frac{\sin v^4-3\sin v^3+3\sin v^2-\sin v+4\sin v^5-12\sin v^4+12\sin v^3-4\sin v^2}{(\sin v-1)^3}=$$

$$=\frac{(\sin v+2^2\sin v^2)(\sin v^3-3\sin v^2+3\sin v-1)}{(\sin v-1)^3}=\sin v+2^2\sin v^2,\text{ quare}$$

fit: $\sin v+2^2\sin v^2=\sin v+2^2\sin v^2$

Simili modo pro $x=3$, $x=4$, $x=5$ etc. etc. illam veram esse facile demonstrari potest.

Fac denique esse $v=\frac{\pi}{2}$ tum habebimus:

$$6)\ \frac{x^2-(2x^2+2x-1)+(x+1)^2-2}{(1-1)^3}=\frac{0}{0}=1+2^2+3^2+4^2+\ldots+x^2$$

et si fractionis et numeratorem et denominatorem, x constans v vero variabile considerantes, differentiemus, obtinebimus:

$$\frac{x^2(x+3)\sin v^{x+2}\cos v-(2x^2+2x-1)(x+2)\sin v^{x+1}\cos v+}{3\sin v^2\cos v-3\,.\,2\sin v\cos v+3\cos v}$$

$$\frac{+(x+1)^3\sin v^x\cos v-2\sin v\cos v-\cos v}{3\sin v^2\cos v-3\,.\,2\sin v\cos v+3\cos v}\text{ aut}$$

$$\frac{x^2(x+3)\sin v^{x+2}-(2x^2+2x-1)(x+2)\sin v^{x+1}+(x+1)^3\sin v^x-2\sin v-1}{3\sin v^2-3\,.\,2\sin v+3}=$$

$$=\frac{0}{0}\text{ pro }v=\frac{\pi}{2};$$

quodsi iterum differentiemus, prodit:

$$\frac{x^2(x+3)(x+2)\sin v^{x+1}-(2x^2+2x-1)(x+2)(x+1)\sin v^x+}{3\,.\,2\sin v-3\,.\,2}$$

$$\frac{+(x+1)^3x\sin v^{x-1}-2}{3\,.\,2\sin v-3\,.\,2}=\frac{0}{0}\text{ facto } v=\frac{\pi}{2}$$

atque iterum differentiantes,

$$\frac{x^2(x+3)(x+2)(x+1)\sin v^x-(2x^2+2x-1)(x+2)(x+1)x\sin v^{x-1}+}{3\,.\,2\,.\,1}$$

$$\frac{+(x+1)^3x(x-1)\sin v^{x-2}}{3\,.\,2\,.\,1}=\frac{2x^3+3x^2+x}{6}$$

sumto $v=\frac{\pi}{2}$ nanciscimur, quae est revera summa seriei superioris (6).

Pari quoque ratione series secundum potestates cosinus procedentes pertractare possemus, quae ubi $v=\pi$ positum esset, ad series nos alias adducerent, quarum termini essent iterum alternis signis ornati: sed quum methodum magis, qua variae summentur series, proponere nostra interesset, quam momenta, quaecunque observari possunt, explorare funditusque perscrutari; neque finem quoque libelli nostri transgredi possimus, pauca solum commemoraturis in eo consistere nobis liceat.

§. 8.

Sit $n=2$ aequatio primitiva 1 §. 1 abibit in hanc:

$$1)\ A_2f(x+2h)+A_1f(x+h)+Af(x)=\varphi(x)$$

atque obtinebimus ob 2 §. 1

$$a_0=A+A_1+A_2$$
$$a_1=A_1+2A_2$$
$$a_2=(A_1+2^2A_2)\frac{1}{1.2}$$
$$a_3=(A_1+2^3A_2)\frac{1}{1.2.3}$$

. . . . .

. . . . .

$$a_k=(A_1+2^kA_2)\frac{1}{k!}$$

et igitur seriem quaesitam:

$$(A+A_1+A_2)f(x)=\varphi(x)+\alpha_1h\varphi'(x)+\alpha_2h^2\varphi''(x)+\dots$$

ubi $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ etc. ex 5 §. 1 aequationibus:

$$A_1+2A_2+\alpha_1(A+A_1+A_2)=0$$

$$\frac{A_1+2^2A_2}{1\,.\,2}+\alpha_1(A_1+2A_2)+\alpha_2(A+A_1+A_2)=0$$

$$\frac{A_1+2^3A_2}{1.2.3}+\alpha_1\left(\frac{A_1+2^2A_2}{1\,.\,2}\right)+\alpha_2(A_1+2A_2)+\dots=0$$

. . . .

. . . .

$$\frac{A_1 + 2^k A_2}{k!} + \alpha_1 \left(\frac{A_1 + 2^{k-1} A_2}{(k-1)!}\right) + \ldots + \alpha_k (A + A_1 + A_2) = 0$$

secundum jam notam legem generalem determinantur. Quodsi iterum casum promere volumus, ubi $f(x) = \Sigma^2 \varphi(x)$ erit, aequationem 1^{mam} in aequationem differentiarum commutare debemus, quae quum secundum 2 §. 3 posito $n = 2$ sit:

$$Bf(x) + B_1 \Delta f(x) + B_2 \Delta^2 f(x) = \varphi(x),$$

facto ergo $B = 0$ atque $B_1 = 0$ prodit:

$$B_2 \Delta^2 f(x) = \varphi(x)$$

$$f(x) = \frac{1}{B_2} \Sigma \varphi(x),$$

si B_2 nullam continet quantitatem variabilem.

At est tum quoque $a_0 = 0$, $a_1 = 0$ (ibidem) quare obtinebimus:

$$\Sigma^2 \varphi(x) = \frac{1}{h^2} \int^2 \varphi(x) dx^2 + \frac{\alpha_1}{h} \int \varphi(x) dx + \alpha_2 \varphi(x) + \alpha_3 h \varphi'(x) + \ldots$$

Coëfficientes vero $\alpha_1, \alpha_2, \alpha_3 \ldots$ ex 24 §. 3 definiuntur aequationibus his:

$$\frac{\Phi_2^3}{1.2.3} + \alpha_1 = 0$$

$$\frac{\Phi_2^4}{1.2.3.4} + \alpha_1 \frac{\Phi_2^3}{1.2.3} + \alpha_2 = 0$$

$$\frac{\Phi_2^5}{1.2.3.4.5} + \alpha_1 \frac{\Phi_2^4}{1.2.3.4} + \alpha_2 \frac{\Phi_2^3}{1.2.3} + \alpha_3 = 0$$

$$\frac{\Phi_2^6}{1.2.3.4.5.6} + \alpha_1 \frac{\Phi_2^5}{1.2.3.4.5} + \alpha_2 \frac{\Phi_2^4}{1.2.3.4} + \alpha_3 \frac{\Phi_2^3}{1.2.3} + \alpha_4 = 0$$

$$\vdots$$

ergo: $\alpha_1 = -1$

$$\alpha_2 = +\frac{5}{12}$$

$$\alpha_3 = -\frac{1}{12}$$

$$\alpha_4 = +\frac{3}{720}$$

$$\vdots$$

Sit e. g. $\Sigma^2 \varphi(x) = 1.2.3 + 2.3.4 + 3.4.5 + \ldots x(x+1)(x+2)$, quum incrementum $= 1$ ponatur, erit:

$$\Sigma \varphi(x) = (x+1)(x+2)(x+3)$$

$$\varphi(x) = 3x^2 + 15x + 18 \text{ quare fit}$$

$$\int \varphi(x) = x^3 + \frac{15}{2} x^2 + 18x + C$$

$$\int^2 \varphi(x)dx^2 = \frac{x^4}{4} + \frac{15}{6}x^3 + 9x^2 + Cx + C'$$

$$\varphi'(x) = 3 . 2x + 15$$
$$\varphi''(x) = 3 . 2 . 1$$
$$\varphi'''(x) = 0$$

Itaque habebimus: $\Sigma^2 \varphi(x) = \frac{x^4}{4} + \frac{3}{2}x^3 + \frac{11}{4}x^2 + \left(C - \frac{49}{4}\right)x$ $+ C' - C + \frac{251}{40}$

Quoniam autem pro $x = 0$ est $\Sigma^2\varphi(x) = 0$, pro $x = 1$ vero $\Sigma^2\varphi(x) = 6$, ad definiendas constantes odtinebimus aequationes:

$$0 = C' - C + \frac{251}{40}$$

$$6 = C' - \frac{177}{120}$$

ex quibus prodit:

$C = \frac{165}{12}$, $C' = \frac{299}{40}$ qui valores si recipiantur, summam completam:

$\Sigma^2 \varphi(x) = \frac{x^4}{4} + \frac{3}{2}x^3 + \frac{11}{4}x^2 + \frac{3}{2}x$ nanciscimur.

Si omnino $\Sigma^2\varphi(x) = F(x) + Cx + C'$ inventum est, atque pro $x = a$ $\Sigma^2\varphi(x) = A$ et pro $x = b$, $\Sigma^2\varphi(x) = B$ proficiscitur, tum definiendis constantibus:

$$A = F(a) + Ca + C'$$
$$B = F(b) + Cb + C'$$

conveniant aeqnationes necesse est, ex quibus:

$$C = \frac{A - B - F(a) + F(b)}{a - b}$$

$$C' = \frac{aB - bA + bF(a) - aF(b)}{a - b}$$

constituitur, itaque summa completa:

$$\Sigma^2\varphi(x) = F(x) + \frac{A - B - F(a) + F(b)}{a - b}x +$$
$$+ \frac{aB - bA + bF(a) - aF(b)}{a - b} \text{ erit.}$$

Quodsi pro B_2 functionem quoque ipsius x poneremus alias nihilominus atque in praecedentibus paragraphis magni momenti inveniremus series.

Quomodo res se habeat, si $n = 3$, $n = 4$, $n = 5$ etc. fuerit, ex his, quae jam instituimus, atque tractavimus, plane intelligitur.

Postremo commemorare quoque debemus, formulam 2 §. 3 summandis seriebus expeditissime accommodari posse. Dato enim f (x) atque $A, A_1, A_2, A_3 \ldots$ secundum quamlibet legem conformatis, summa φ (x) seriei 1, §. 1, facile invenitur, siquidem quaevis differentia ipsius f (x) (2, §. 3)

est constans, quam ultra nulla sequitur alia. Prout iterum x incrementum h atque quantitates A, A_1, A_2, A_3 etc. (in 1 §. 1) aliter determinantur, aliae multae quoque derivantur series, quae summari possunt, si summa seriei generalis inventa et definita est.

Vita.

Natus sum Josephus Theophilus Ustymowicz fidei catholicae addictus d. IV. Jul. anno hujus seculi vicesimo, vico, cui nomen est Graboszewko, prope oppidum Września sito, patre Floriano, ante hos septem annos mihi lugubri morte erepto, matre vero Nepomucena e gente Sypniewska, cujus decessum jam ab anno sexto aetatis meae dolenter quoque lugeo. Primis literarum elementis domi imbutus gymnasii Tremesnensis, Meissnero tunc rectore clarissimo, nunc vero illustrissimo Dziadek florentis disciplinae traditus sum, ubi a. MDCCCXLII examine, quod abeuntium vocatur, legitime superato, maturitatis testimonio praeditus unum post semestre Vratislaviam me contuli mathematicis praecipue studiis operam navaturus. Hic ab Ill. Benedicto t. t. rectore magnifico inter cives academicos rite receptus, Ill. Fischero t. philosophorum decano spectatissimo apud philosophos nomen dedi, quorum scholis quatuor annos et dimidium interfui. Ac mathesin primum docuerunt me: V. V. Cll. DD. Kummer, Koch et Rosenhain; physicen Gravenhorst, Pohl, Goeppert, Glocker, de Boguslawski, Duflos, Müller et Kenngott; historiam et geographiam Stenzel, Kutzen et Roepell; philosophiam Thilo et Braniss; philologiam Elvenich; historiam denique literarum germanicarum Kahlert, Jacobi, Freytag, slavonicarum vero Celakowsky. Quibus omnibus viris doctissimis atque optime de me meritis, qui rectam studendi viam benignissimo semper animo mihi monstrarunt, omnes studiorum fructus, quoscunque percepi, debere, atque quam maximas possum gratias habere semperque me habiturum, palam profiteor.

Theses.

1. Eo perfectior omnis scientia, quo magis ad leges matheseos redigi potest.
2. Signum aequationis non semper unum idemque significare contendo.
3. Errant, qui partem physices, quae dicitur mathematica, a physice omnino separari velint.
4. Generatio primitiva seu aequivoca in rebus naturalibus non est ponenda.
5. Sententia, naturam saltum non facere, de omnibus rebus probari nequit.